Mathematicians and Mathematics Textbooks for Prospective Elementary Teachers

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Over the last decade, mathematicians and mathematics educators have turned their attention to teacher knowledge, responding to standards documents and state policies (Wilson, 2002) and to disappointing results of various national and international assessments of U.S. student performance in mathematics. It is the purpose of this article to explore the approach taken in four mathematics textbooks, written by research mathematicians for courses for prospective elementary teachers, to the problem of what teachers need to know and how they should learn it.

In 2001, the Conference Board of the Mathematical Sciences (CBMS, 2001) made recommendations about the mathematics K–12 teachers need to know. In the last decade, researchers in mathematics education have begun to specify and assess teacher knowledge in new ways (Ball, 2003; Hill et al., 2004; Ma, 1999) and research mathematicians have become increasingly involved in teacher education and the professional development of mathematics teachers. For example, Milgram (2004) gives a detailed account of a proposed series of courses that differ significantly from the CBMS guidelines, addressing what he thought were important shortcomings of the CBMS report. Four research mathematicians have recently written complete (Beckmann, 2003; Jensen, 2003; Parker and Baldridge, 2003) or partial (Wu, 2002) textbooks for mathematics courses for prospective elementary teachers.[1]

While there is a general expectation that elementary teachers know enough mathematics to teach it, there are differences of opinion about what should be included in courses designed for elementary teachers, and how it should be taught (Askey, 1999; Ball, 2003; Ball and Bass, 2003; CBMS, 2001; Hill et al., 2004; Ma, 1999). One way to begin to analyze *what* is taught is to look at the mathematics textbooks written specifically for courses for prospective elementary teachers. In classes that use these textbooks, the books define a substantial element of what students have an *opportunity* to learn. Many states or certifying institutions require one, two, or more semesters of mathematics to qualify for an elementary teaching certificate. Some institutions require a minimum number of mathematics courses from the undergraduate curriculum, while others require classes specifically designed for elementary teachers. These latter are usually taught in mathematics departments. Certifying institutions may also require mathematics methods courses, usually taught in departments of education. It is textbooks for *mathematics* courses for elementary teachers to which our attention is turned.

Twenty textbooks, including the four recently written (or in preparation) by mathematicians, are being analyzed as part of a project aimed at investigating the mathematical education of prospective

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[1] *There are other textbooks in preparation to which I did not have access.*

elementary teachers.[2] In the following sections, I first give a general overview of how the books can be characterized as textbooks and how the mathematicians' books differ from previous texts; then I look specifically at one topic—the definition of fractions—for a detailed view of some of the mathematical issues entailed in teaching elementary teachers. The focus will be on the four mathematicians' books, with some reference to other texts for particular contrast. Specifically, I explore the following questions:

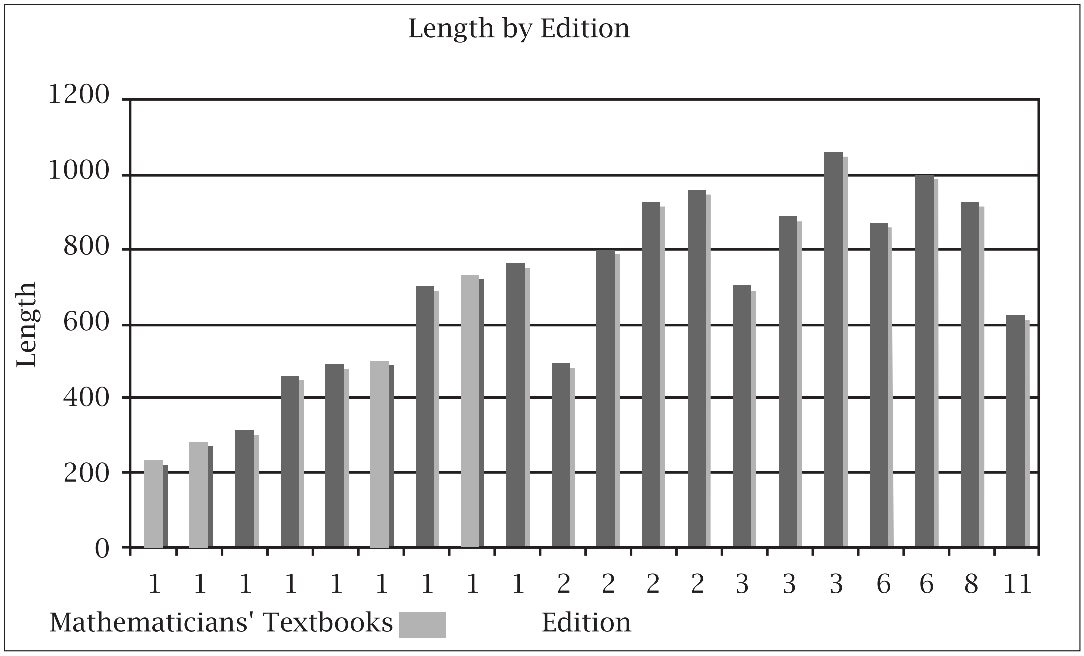

Figure 1. Textbook length by edition.

1. How can these textbooks be characterized? In what ways are the mathematicians' textbooks different from the previous generation of books (written primarily by mathematics educators)?

2. What complexities of teaching mathematics and of teacher knowledge for such teaching are reflected in these four books?

Characterizing the Books

Analysis of the twenty books suggests that while there are many similarities in topics and overall content, they differ substantially in how they approach the mathematics elementary teachers need to learn. I have identified four aspects of the books that provide a framework for distinguishing among them: type of book, coherence, rigor, and claims.

Type of Book

In the overall analysis of the twenty textbooks, it is clear that the books differ in their purpose and design. Most of the books are encyclopedic, including every possible topic that might be covered in K–8 classrooms, and treating each topic as a separate entity (e.g., Billstein et al., 2003). These books are long, well indexed, and comprehensive. They typically have supplementary materials such as activity manuals, websites, and CD-ROMS. Other books are shorter and more concise, not aiming to cover all bases, but rather giving emphasis to some topics while mentioning others only briefly (e.g., Parker and Baldridge, 2004). These books tend to be more narrative in their approach to mathematics, in the sense of telling a nuanced story of mathematics with big ideas in focus as the leading characters. In between are books that may "cover" all the topics but emphasize some more than others, or books that take a particular approach to the mathematics. For example, Masingila (2002) takes a problem-solving approach where the mathematics is introduced through problems. On this scale from encyclopedic and comprehensive to narrative and focused, the books by mathematicians are toward the narrative, focused end. This is not to say that they leave out important mathematics; rather, they provide a mathematical landscape where some topics and approaches are clearly more important than others.

On the comprehensive end of the scale, books serve as a resource for a class rather than a recipe for a course or a sequence of courses. Although it is at least conceivable that an instructor could completely "cover" an encyclopedic book in several semesters, he or she will likely need to pick and choose among topics, problems, and activities to teach a course from one of these books. On the other end of the scale, some of the books were written specifically to define a course (or sequence). I include here not only three of the books by mathematicians, but also books by other authors (e.g., Darken, 2003; Masingila 2002). Several authors (Darken, Masingila, Baldridge, Parker and Beckmann, personal communications) have suggested in conversations and interviews that they wrote the books to use in their own teaching *in response to* what they saw as a need for a different textbook.

One wonders if the books by mathematicians are shorter and more narrative because the authors are mathematicians or for other reasons. Other explanations are possible. For example, nearly all of the encyclopedic books are in multiple editions,

[2] *For a complete list of books, go to* http://www.msu.edu/~mccrory/textbooks.htm. *Three books on the list have gone out of print since January 2005, and they are noted on the Web site. For more information on the larger project, go to* http://www.educ.msu.edu/Meet/.

having been originally published many years ago. They are products of major publishing houses that may have influenced the content and layout of the books in ways that push them toward comprehensiveness and toward a particular editorial style. In fact, comparing book length to number of editions (Figure 1), suggests a regular pattern of increasing length with increasing editions, with a few exceptions. The mathematicians' texts are highlighted, all in first edition and all on the shorter side.

The difference pointed to here—between encyclopedic and narrative books—goes beyond length. In the introduction to her book, Beckmann writes:

> The book focuses on explaining why. Prospective elementary school teachers will learn to explain why the standard procedures and formulas of elementary mathematics are valid, why nonstandard methods can also be valid, and why other seemingly plausible ways of reasoning are not correct.... [T]eachers will come to organize their knowledge around the *key concepts and principles* of mathematics, so that they will be able to help their students do likewise. (p. ix, my emphasis)

Her intention is to organize the book around key ideas, and her book proceeds with an exposition of arithmetic built around operations. Jensen takes a different approach, organizing his book "in the old-fashioned style of definition, theorem, proof used in Euclid's *Elements*" (p. viii). Parker and Baldridge tie their textbook directly to a textbook series for elementary students (the Singapore series), focusing on helping prospective teachers learn one clear and logical development of elementary mathematics, rather than covering every possibility. Wu writes that his book "says only what needs to be said, so you will have to read every line and try to understand every line. This monograph tells a coherent story, but the outline of the plot (the procedures) is already familiar to you. It is the details in the unfolding of the story (the reasoning) that are the focus of attention here" (chapter 1, p. 2).

Coherence

The second characteristic of the new textbooks that distinguishes them as a group is their mathematical coherence. This is closely related to the type of book, yet distinctive enough to merit separate consideration. In each of these three complete books, and in Wu's partial book, the authors take a perspective on the mathematics that yields a *sense of mathematics as a discipline*.

For example, Beckmann approaches elementary mathematics through operations: rather than build up each number system separately, including the operations within each system, she defines and develops addition and subtraction as the headline topic and develops the number systems within addition and subtraction. Then she does the same with multiplication and finally with division, from whole numbers to rationals. This allows for a focus on operations and their definitions, bringing to the fore how they are the same no matter what kind of numbers are involved. Research on mathematics education has long recognized that students have misconceptions about operations, for example, seeing multiplication of fractions as something different from multiplication of integers (Harel et al., 1994). Beckmann's approach addresses this problem mathematically by providing uniform definitions of operations that apply across number systems.

Wu, in his two chapters, emphasizes the importance of starting from definitions and building a coherent mathematical system from those definitions:

> The way mathematics works is to start with one clearly stated meaning (i.e., a precise definition) of a given concept, and on the basis of this meaning we explain everything that is supposed to be true of this concept (including all other meanings and interpretations) using logical reasoning. (Wu, Chapter 2, p. 9)

He is explicit about mathematical processes and practices, pointing to definitions as basic building blocks and showing how alternate definitions, models, or conceptions flow from the primary definition.

Parker and Baldridge build the mathematics through "teaching sequences" that start with basic concepts accessible to young children and increase in complexity while retaining mathematical integrity. In a way, these two books—Wu on the one hand, and Parker and Baldridge on the other—start at opposite ends of the mathematical terrain. Wu presents concise, accurate and final definitions, working backwards from them to show how other conceptions and models fit. Parker and Baldridge build up from the simple to the complex, in a progression actually found in K–8 classrooms (using the Singapore materials.) Both are coherent, accurate conceptions of the mathematics that give a sense of mathematics as a discipline, and as something that, above all, makes sense.

At the other extreme, a textbook might present mathematics as a collection of topics, each approximately equal in value, that are related by virtue of being called mathematics (or dealing with numbers and symbols). Although none of the twenty textbooks give an *incoherent* picture of mathematics, in some books there is a "flatness" (Cuoco, 2001) that makes it difficult to tell what is important and how the pieces all fit together to make sense.

Rigor

Dictionary definitions of the word "rigor" suggest strictness and severity. But in mathematics, rigor is a virtue: it is correctness, completeness, sparseness and elegance all rolled into one. While implicitly recognizing that the standard of rigor is different from that for peer-reviewed mathematics journals, the mathematicians have written textbooks that aim for and achieve rigorous mathematics. They pay attention to definitions, logical development of topics, making connections across topics, and mathematical reasoning. These are *mathematics* textbooks in ways that some of the other books, written by nonmathematicians, are not.

Jensen's book is perhaps the paradigmatic version of the rigorous mathematics textbook. He takes a definition/theorem/proof approach to the entire subject. Every procedure and algorithm is proved, starting from basic definitions. The other mathematicians' books include some proofs, and all of them emphasize the importance of clear, consistent, and correct justification. They are often (although not always) explicit in trying to teach the prospective teachers about the importance of rigor and clarity in mathematics, portraying mathematics as an endeavor in which care and accuracy are both important.

In all four books, the authors emphasize, explicitly and implicitly, the importance of definition; of building from definitions to other representations, models, or alternative definitions; and of mathematics as a subject that makes sense. By contrast, in one of the encyclopedic textbooks, the author presents several alternative conceptions of fractions without providing a starting definition, and without showing how the different conceptions relate to each other and, in fact, define the same mathematical object. In the books by mathematicians, such mathematical sloppiness does not occur.

Claims

Finally, with the exception of Beckmann, the mathematicians make definitive claims in their books about the right ways to teach mathematics. Implicitly, every book makes such claims by virtue of its contents and rhetoric, but in these books, the claims are explicit.[3]

Where do their claims come from? All of these mathematicians have many years of teaching experience, and of course, many years of experience as mathematicians. They have all worked with prospective elementary teachers (and some have provided professional development to practicing teachers). Thus, they draw on their own experience—as teachers and as mathematicians. Milgram (2004) in his recommendations for courses uses a program developed in Russia in the 1930s and 1940s that made its way to Israel and later China. This program was adopted for use in the Singapore texts (Hong, 1999), which Parker and Baldridge use extensively. "[W]e pay a great deal of attention to the way in which the Russian program develops the core concepts in mathematics during the early years, and we also reference the Singapore program extensively to learn about how the three topics [Shulman's types of mathematical knowledge for teaching] are treated in countries where instruction in mathematics is successful" (Milgram, p. 9). Parker and Baldridge's book is used in conjunction with five of the Singapore booklets and includes homework assignments in these books as well as examples taken directly from them. The argument of these authors is reasonable enough: we can deduce from successful programs what it takes to be successful, and we can extend that from teaching elementary students to teaching elementary teachers. Milgram writes,

> The emphasis on precision of language and definitions matters most for exactly the most vulnerable of our students. It is these students who must be given the most careful and precise foundations. The strongest students often seem able to fill in definitions for themselves with minimal guidance. On the other hand, foreign outcomes clearly show that with proper support along these lines, all students can get remarkably far in the subject. (p. 10)

The logic of this argument is compelling, but it is a logical argument, not an empirical one. Research in mathematics education has made substantial contributions to our understanding of how children learn, what misconceptions they are likely to have, and other individual or psychological aspects of mathematics learning. More elusive in empirical research is how to turn these findings about learning mathematics into successful teaching across a wide range of students and teachers.[4]

Whatever their basis, the mathematicians make claims in ways that are not found in other books. Again, this may not be because they are mathematicians: the books with the most claims about teaching and learning—the Wu chapters, the Parker

[3] *Note that claims and recommendations about teaching are common in instructor's manuals that accompany other textbooks. The distinction here is that, in these books, the claims are an integral part of the mathematics text itself.*

[4] *It would be particularly interesting to know whether teachers who learn mathematics in the ways suggested by the mathematicians' textbooks are more successful as mathematics teachers. This is exactly the kind of research that is missing, in part because it is very difficult to accomplish.*

and Baldridge textbook, and the Milgram report—are all books that have avoided the editorial processes of a major publisher. One can imagine that editors deliberately depersonalize textbooks, taking out all kinds of personal ideas and even the "voice" of the author, especially as books are revised and reissued over time. And one can imagine that textbook publishers want to avoid controversial statements and edit out anything that touches on opinion.[5]

What are some of these claims? The following are taken from the books' introductions and sections on fractions.

> The best way to (develop an understanding of elementary mathematics at the level needed for teaching) is to study actual elementary school textbooks and to do many, many actual elementary school mathematics problems. (Parker and Baldridge, p. viii)

> Spoken statements such as '2 fifths + 2 fifths = 4 fifths' and '2 sevenths + 3 sevenths = ___ [sic] sevenths' are immediately clear to children. (Parker and Baldridge, p. 135)

> $\frac{k}{l} + \frac{m}{n} = \frac{kL+mN}{A}$ where $A = nN = lL$ (11)
>
> The worst possible abuse is the use of (11)—with A as the lcm of n and l—as the definition of the addition of the two fractions $\frac{k}{l}$ and $\frac{m}{n}$ … it is enough to point out that formula (11) is a pedagogical disaster when used as the definition of adding fractions…. Therefore there is no contest: one should never teach fractions using (11) as the definition of addition. (Wu, Chapter 2 p. 53)

> Fractions are best introduced to students using dollars and cents, since these are objects of intense interest. … Students should next be introduced to the area model for fractions… (Milgram, p. 27–28)

> For these students [preservice elementary teachers], the key to learning this material is getting all the facts, expressed at the right level, with all the details, and at a pace slow enough to allow proper absorption. (Jensen, p. viii)

While none of these statements is necessarily false, they are each a kind of claim rare in other textbooks. A collection of such statements, taken from the books written by these mathematicians, could well define a research agenda for mathematics education. The rhetoric in these texts sometimes suggests that we know such things as empirical facts, while these assertions are more likely the opinions of the authors, arrived at through a combination of logical mathematical analysis of the topics, personal experience, and in some cases research from mathematics education or other third-party evidence.

Whether the claims made are true or false, justified or not, the point here is that the other books —by non-mathematicians—are less likely to contain these kinds of claims. It is possible that this is a rhetorical style familiar to mathematicians and that their "claims" are meant to be taken the way one takes mathematical claims—as subject to further consideration in the face of different assumptions or new evidence, but always stated in their strongest, most cogent form, and logically justified.

Unfortunately, we do not have conclusive empirical evidence about the best ways to teach or about the best ways for all students to learn. In fact, empirical studies are fraught with problems that make the possibility of drawing unqualified conclusions about "the best" ways to teach and learn unlikely. Such conclusions depend on too many uncontrollable variables: the particular students in the class and their particular mathematical backgrounds, the resources available in the classroom, the number of students, the grade level, etc. The assumptions one would need to make, the definitions of key ideas, and the "axioms" on which such deductions could be based would be extremely restrictive.[6] Even if one could specify an "ideal" class, there is still the question of how we could know that a particular approach to a topic is best. As we will discuss below, even mathematicians do not agree on some very basic aspects of teaching mathematics.

Complexities

One of the interesting aspects of the mathematicians' books is that they sometimes disagree about important mathematical ideas. At these points of disagreement, we have insight into some of the complexities of teaching mathematics to

[5] *On the other hand, Beckmann (personal communication) reports that her publisher did not intervene in the content of her book in any way, suggesting no changes in style or content.*

[6] *For example, to draw conclusions about "the best" way to teach a topic, one might need to specify class size, teacher qualifications, students' prior knowledge and preparation, length of the class period, characteristics of the (ideal) textbook or curriculum materials, and more. For each of these, definitions and valid measurement would be required.*

elementary teachers. It is possible that if all these authors were in the same room they could agree about how to approach this mathematics, and of course, there is no mandate that there must be agreement on all things mathematical. But as written, the mathematics is different from book to book in ways that might matter to prospective elementary teachers. Here are some of the points on which the books differ, posed here as questions (taken from sections on the definition of fractions in each book). Keep in mind that these are textbooks addressed to elementary *teachers*, not to children.

1. How are fractions best defined? How is that choice of (primary) definition justified mathematically and/or pedagogically?

2. Should a distinction be made between fractions and rational numbers? If so, exactly how should each be defined?

3. Should a distinction be made between the concepts of equal and equivalent fractions?

4. Should teachers know alternative definitions of fractions? Part/whole? Set theory? Number line? Division? How can equivalence among different definitions or models best be illustrated and taught?

5. Is it important to distinguish between a symbol and what it stands for? If so, how do we do that for fractions?

6. Is it important to have a single definition for a mathematical concept—like fraction—and use it exclusively throughout the text? To what extent, and in what ways, can a definition change within the text? What terms or concepts can be used in a definition—that is, what can be taken as given in a definition?

7. What do teachers need to learn explicitly about the role of definition in mathematics, and what can (or should) be left implicit?

8. Do these students need to do formal proofs of things like rules for fractions? What do they need to understand about proofs in general?

9. Is it better to teach prospective teachers a single approach to fractions, or teach them all the different approaches they might see in the curriculum materials they are confronted with as teachers? Is the latter the purview of a mathematics course, or of a methods course?

The mathematicians' books—as well as the other books—differ in important ways with respect to these questions. In the next section, I illustrate differences in the definition of fractions, and discuss why it might matter that mathematicians do not come to the same conclusions when they think hard—and apply their own extensive experience as teachers and mathematicians—about the best ways to teach elementary mathematics to elementary teachers.

An Example: Defining Fractions

Three of the mathematicians' textbooks and the Milgram book define a fraction as a point on the number line with particular characteristics. One includes both a part/whole definition and a number line definition. One uses only a part/whole definition. Among the other sixteen books, the most common definition is similar to that in Billstein, et al (2003):

> [N]umbers of the form a/b are solutions to equations of the form $bx = a$. This set, denoted Q, is the set of rational numbers and is defined as follows:
>
> $Q = \{a/b \mid a$ and b are integers and $b \neq 0\}$ (p. 266)

Although this may be a legitimate definition (depending on what has been previously defined), it is problematic: it assumes a definition of "number" and "equation"; it assumes knowledge of multiplying a fraction by a whole number; and it is quite removed from any definition that a teacher would be likely to use with a child.

What do the mathematicians do? The definitions given below are what the authors explicitly call the definition of fractions—all of the authors use other models in their complete exposition of fractions, but build on a fundamental definition. They all agree on, and make explicit, the importance of providing precise and rigorous definitions, not only for fractions but throughout their texts. The differences in their definitions raise interesting questions about what prospective teachers need to know and how they can learn it, as well as what a textbook author can assume in stating definitions.

Consider, for example, Milgram's definition of fractions (Milgram, p. 222):

> [Positive fractions] will be numbers of the form $\frac{a}{b}$ where a, b, are whole numbers and $b \neq 0$, and their definition is as follows. Divide the segment from 0 to 1 into b equal parts, which in this context means b non-overlapping congruent subsegments (here "congruent" simply means two sub-segments can be made to coincide completely by sliding one on top of the other). Do this for the segment between 1 and 2, between 2 and 3, and so forth. These divisions create a special collection of points, namely, the totality of the endpoints of these smaller segments. The leftmost of these division points is 0, and the rest of them form an equi-spaced collection to the right of 0 and they include whole numbers. We now give names to these division points: starting with 0, the first

> one to the right of 0 will be $1/b$. The second $2/b$, the third $3/b$, etc. In general, if a is any nonzero whole number, $\frac{a}{b}$ is the a-th of these division points to the right of 0…. The number $\frac{a}{b}$ is called *the fraction with numerator a and denominator b*.

In the middle of this is a definition of congruence that uses the idea of "sliding". It gives a mathematically correct, clear image of what it means for two segments to be congruent. One might wonder, however, whether these undergraduate students would understand why it is acceptable to use Milgram's explanation of congruence: "'congruent' simply means two sub-segments can be made to coincide completely by sliding one on top of the other." For these students, this definition may create the impression that anything goes in a definition. It is mathematically quite sophisticated to know when such an explanation is acceptable as part of a definition (cf., Lakatos, 1987). If that is okay, why is it wrong to define congruent triangles in a similar way, or to prove congruence by cutting out the objects and placing them on top of one another? How is this different from proof by example? In geometry and other high school mathematics courses, students learn that examples do not prove; yet this definition seems, at a naïve level, to be based on an example. What is taken as given? What is already defined? It is not that his definition of fraction is unclear, but rather that it raises the question of what is mathematically acceptable in a definition, and how students can learn what constitutes a mathematically acceptable definition.

Wu's definition creates similar issues (Wu, p. 4):[7]

> **Definition.** Let k, l be whole numbers with $l > 0$. Divide each of the line segments [0,1], [1,2], [2,3], [3,4],… into l segments of equal length. These division points together with the whole numbers now form an infinite sequence of equally spaced markers on the number line (in the sense that the lengths of the segments between consecutive markers are equal to each other). The first marker to the right of 0 is by definition $\frac{1}{l}$. The second marker to the right of 0 is by definition $\frac{2}{l}$, the third $\frac{3}{l}$, etc., and the kth is $\frac{k}{l}$. The collection of the $\frac{k}{l}$'s for all whole numbers k and l, with $l > 0$, is called the *fractions*. The number k is called the *numerator* of the fraction $\frac{k}{l}$ and the number l its *denominator*.

[7] *Wu also uses the idea of "sliding" to determine the length of a segment and to check for congruence (e.g., Chapter 2, pp. 7 & 11). An anonymous reviewer pointed out, and Wu confirmed, that their two definitions are not independent. They worked together on the committee that led to Milgram's book.*

What does it mean to say: "These division points together with the whole numbers now form an infinite sequence of equally spaced markers on the number line (in the sense that the lengths of the segments between consecutive markers are equal to each other)"? Where do these new terms—infinite sequence, equally spaced markers, length of segments—come from and how do we know what they mean? Can we assume that a prospective elementary teacher will understand not just the words, but why they can be used as part of a rigorous definition of fractions? This is not to say that the definition is incorrect or ambiguous (although it could be ambiguous or even meaningless to someone with little mathematical background), only that the words seem rather magical. The textbook author knows when he can use undefined terms or physical analogies and which words and ideas can be taken as given, but does the student?

Beckmann uses a part/whole definition and first defines a fractional *quantity* (Beckmann, p. 58):

> If A and B are whole numbers, and B is not zero, then the *fraction* $\frac{A}{B}$ *of an object, a collection, or a quantity* is the amount formed by A parts (or A copies of parts) when the object, collection, or quantity is divided into B equal parts.

She emphasizes use of the word "of" to call attention to the importance of the unit:

> Notice the crucial word *of* in the examples of fractions of objects, fractions of collections of objects, and fractions of quantities… Fractions are defined in *relation to a whole*, and this whole can be just one object, or it can be a collection of objects, such as the cars on the road, or 24 houses. …Students from elementary school through college can correct many mistakes in their work with fractions if they can identify the whole associated with a fraction. That is, they need to understand what the fraction is *'of'*. (p. 59, emphasis in original)

This is an essential part of her definition, which she uses throughout the chapter on fractions. Later in the book, to define fractions as numbers on the number line, Beckmann begins with the following:

> We create the notion of the whole numbers by abstracting from our experiences with objects. For example,
>
> 2 apples, 7 balls, 25 people,…

is really like saying (somewhat awkwardly)

2 of apple, 7 of ball, 25 of person,…

which abstracts to the following notion of number: 2, 7, 25,… In the same way, we create the following notion of fractions as numbers by abstracting from fractions of objects: 2/3 of a pie… [abstracts to] 2/3… but even when fractions are viewed abstractly as numbers, they are still "of a whole". Just as 5 is "five ones," so, too, ____ is "3/4 of 1". (p. 77)

Insisting on consistency and use of the definition of fractions in defining all aspects of fraction arithmetic, Beckmann generates these admittedly awkward constructions, "2 apples" means "2 of apple", to reach the conclusion that a fraction is a number on the number line, the same way that whole numbers are numbers on the number line.

Jensen's definitions (he uses two definitions to define fractions and their values) are as follows (Jensen, pp. 91 & 190):

Definition 2.104. The *fraction* $\frac{m}{n}$ of an object is the amount obtained by dividing the object into n equal parts and taking m of these parts.

Definition 5.1. The fraction $\frac{p}{q}$ represents the point on the number line arrived at by dividing the unit interval into q equal parts and then going p of these parts to the right from 0. This point is called the value of the fraction. A *rational number* is the value of some fraction.

Is the distinction between a point on a number line, the value of that point, and the fraction it defines something a teacher needs to understand? None of the other mathematicians' books make this distinction. Instead, when using the number line, they treat fractions as numbers or points on the number line, and equivalent fractions as different names for the same numbers or points. Which is correct? Does it matter for elementary teachers?

Parker and Baldridge take a different approach, defining a fraction as a point on a number line, but with a relatively intuitive definition, then building up to that definition through the numerous models and definitions that appear in elementary curricula (Parker and Baldridge, p. 131):

A fraction is a point on the number line. For example, to locate 7/5, we start at 0, find the step size so that 5 equal steps gets us to 1, and then take 7 such steps, landing at the points called $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, … until we get to $\frac{7}{5}$.

[An illustration of the number line divided into fifths up to seven-fifths follows.]

Following this definition, they begin with a part/whole model of fractions and work through a "teaching sequence" that leads to the conception of fraction as number. They too use a physical metaphor—creating equal steps—to envision a fraction as a point on the number line. Although the above definition is the first sentence in their chapter on fractions, it is not labeled "definition", and they do not have a place where they specifically designate a definition for fraction.

These differences in how fractions are defined may seem insignificant to some, and it is possible, perhaps likely, that each of these mathematicians would judge the others' approaches as correct even if not ideal. Yet the details and how they are addressed represent sophisticated mathematical issues and point to a fundamental mathematical problem that is replayed across the curriculum: How do we create definitions and starting assumptions that are both mathematically correct and at the same time comprehensible and unambiguous to this population of students (prospective elementary teachers)? Definitions require terms, and terms require definitions. Where do we start with students who may be mathematically unsophisticated at best? This is not a new problem. Mathematics at every level demands attention to undefined terms and first principles. What is new here is balancing the desired rigor of mathematics with the background knowledge of prospective elementary teachers to create a coherent, rigorous, and comprehensible mathematics curriculum for their mathematical education. These authors address the fundamental problem that elementary teachers themselves face: as elementary school teachers must connect children's naïve conceptions of mathematics to mathematics that is correct and comprehensible, so instructors of elementary teachers must connect not only to their own students' conceptions (and misconceptions) of mathematics, but also to the mathematics they are likely to teach to their K–8 students. The mathematical question for textbooks authors, and course instructors, is dual: 1) What is a correct mathematical approach to fractions (or some other topic); and 2) what does an elementary teacher need to know that will allow her correctly and rigorously to build a bridge between that mathematics and what a child can understand?

From the perspective of these students—undergraduates, nineteen or twenty years old whose mathematics background consists of three or so years of high school mathematics—these issues and the differences across these texts can be extremely confusing. Students arrive at their undergraduate mathematics courses with ideas about fractions based on their own elementary and secondary education. They have all seen definitions of fractions and rational numbers, probably multiple definitions, before they take these courses. Whatever definition they are taught needs to somehow cohere with, or correct, their prior knowledge, and help them understand fractions in a way that can be used with children (Ma, 1999). The same issue arises in every aspect of elementary mathematics. We want these students to develop what Ma calls "profound understanding of fundamental mathematics", and yet it is not clear that every approach (in this example, every definition of fractions) makes an equal contribution to such understanding.

Conclusions

These recent books by mathematicians provide important insights into the mathematical education of teachers. In each book, the rigor and coherence, the careful approach to mathematics, the emphasis on definition, the portrayal of mathematics as something that, above all, makes sense combine to provide a view of mathematics as a discipline that is missing from encyclopedic textbooks. Yet these very characteristics create problems that may be inherent in trying to teach a complex, sophisticated subject to naïve learners. The problems with definition of fractions illustrate the complexity of this endeavor, and suggest that we have a long way to go before we reach conclusive answers to the questions of what mathematics we should teach prospective elementary teachers and how it should be presented.

Along with many other mathematicians, some of these authors believe strongly that if mathematics is clearly and correctly explained, prospective teachers can and will learn it. To them, the key is clear, correct, and timely explanations. Although it is certainly true, as Wu points out, that "if students are not taught correct mathematics, they will not learn correct mathematics" (Chapter 2, p. 2), there is no single "correct" version of this mathematics, and we do not know what confusion is generated over time by the small but significant differences in what teachers are taught. If there is any place to quarrel with these books or their authors, it is with claims that there is a single correct way to approach these topics and that the reason teachers have not learned more mathematics in the past is a failure on the part of their teachers to approach mathematics correctly. The lessons from the books by mathematicians are that the mathematics of elementary school has deep and complex roots; that there are different and sometimes conflicting approaches to explaining this mathematics; and that there may be no perfect *mathematical* solution to the problem of how to teach this subject.

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