Making the Abstract Concrete in Mathematics Classes for Future Elementary Teachers

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Abstract

This paper illustrates how three instructors use concrete objects in mathematics classes for future elementary teachers. We argue that using concrete objects with future teachers has purposes that make it mathematically different from teaching K-8 students. The cases show approaches that clarify some of the issues of teaching something abstract (multiplication and division of fractions) using something concrete (a manipulative, picture, or realistic word problem). The study shows that a concrete object may itself become abstract as a substitute for a mathematical idea. We conclude that classes for teachers need to attend to more than learning mathematics through the use of concrete objects; to include *meta* concepts about the mathematics of the objects themselves and how they function mathematically.

Purpose

This paper explores the complexity of using concrete objects to teach mathematics in classes for preservice elementary teachers. Previous research suggests that using concrete objects (including, but not limited to manipulatives expressly designed for teaching) does not simply "tell" students the mathematics they need to learn. Less clear, however, is how concrete objects can be used effectively to teach mathematical content to undergraduate students who will become elementary school teachers. It is the purpose of this study to investigate how instructors use concrete objects and link them to abstract mathematical ideas in mathematics classes for preservice teachers.

Perspective

At least since Piaget, developmental psychologists have held that children develop abstract thinking slowly, starting as concrete thinkers with little ability to create or understand abstractions. For adults, however, there is no question that they are capable of abstract thinking and of using symbols and other representations. Problematic in teaching both children and adults, though, is the fact that, as Anna Sfard puts it, "we [teachers] lose the ability to see as different what children [our students] cannot see as the same": we are blinded by our own knowledge and understanding (Sfard, 2008, p. 59). Mathematicians and teachers of mathematics see that equivalent fractions represent the same number or the same quantity; children and less knowledgeable adults may not see the equivalence of fractions either as numbers or as pieces of pizza.

The idea that children start off as concrete thinkers has been translated into teaching methods that focus on using concrete objects to teach mathematics. There has been considerable research showing that merely using manipulatives to teach mathematics does not guarantee that students will learn (Ball, 1992; DeLoache, 2002; Mix, 2009; Uttal, Liu, & DeLoache, 1999), and little research providing general principles about how best to teach with manipulatives.

The perspective we take in this paper is that concrete objects – tangible things, pictures of such things, and stories about such things – may not be concrete from the perspective of the learner, even an adult learner. We agree with Wilensky who wrote,

> Concreteness is not a property of an object but rather a property of a person's relationship to an object. Concepts that were hopelessly abstract at one time can become concrete for us if we get into the "right relationship" with them. … Concreteness, then, is that property which measures the degree of our relatedness to the object, (the richness of our representations, interactions, connections with the object), how close we are to it, or, if you will, the quality of our relationship with the object. (Wilensky, 1991, no page numbers)

Research on using manipulatives to teach preservice teachers is sparse. In a recent study, Puchner and colleagues (Puchner, Taylor, O'Donnell, & Fick, 2008) provided professional development that included focused attention to teaching with manipulatives. Although their intervention included long-term lesson study groups during which the teachers designed and redesigned lessons, the teachers struggled with using manipulatives in ways that met their learning objectives. In another study, Green and colleagues (2008) provided focused mathematics instruction in a university child development course. They used manipulatives in specific and structured ways (labeled as "guided constructivism" p. 236) for 14 hours of instruction to teach operations on whole numbers and fractions in a pre/post test design. Their students achieved significant gains on all measures, including a measure of their ability to represent correctly the meaning of a fraction division problem. The authors concluded "manipulatives can effectively reverse most arithmetic misconceptions of elementary education majors before they enter classrooms as full-time teachers" (p. 241). Unfortunately, the study does not have any indication of whether the gains were lasting or whether the future teachers could use manipulatives in any ways other than those prescribed in the teaching episodes.

Our framework includes the assumption that using concrete objects to teach mathematics to teachers is different from teaching children because

a. The teachers or future teachers are adults, most likely capable of symbolic and abstract thinking in ways that children are not.

b. They come to the subject with prior knowledge of mathematics, some of it undoubtedly incorrect.

c. They need both to learn mathematics and to learn mathematics for teaching. In particular, they need to learn the mathematics of objects they may use in their teaching.

The last point is particularly important: through the use of manipulatives, they may learn mathematics more deeply, but they also need to learn about the object itself, a kind of meta-level knowledge that their students (K-8 children) *do not* need to learn.

Methods and Data Sources

This study is part of a larger study of mathematics classes for future elementary teachers. Data were collected over a period of two years, including videotaping lessons during which fractions were taught in the classrooms of 7 instructors at 6 institutions. Data from these instructors also include interviews with them during the semester. Data were analyzed in three stages: we created rough transcripts of the video tapes and interviews; we identified relevant episodes; and finally, we created detailed transcripts of episodes and discussed them in the larger group (of 6 researchers) to agree on their meaning in light of our research questions.

Three teachers were selected for this paper, based on the differences in how they used and linked concrete objects and abstractions in their classrooms. Basic information about the teachers and classes is shown in Table 1.

Table 1:

Instructors and courses

	Course	Certification Requirements	Years teaching this course)	Degree & Rank
Dr. Deaver	Mathematics for teachers, 1st in the sequence of 2	2 mathematics, 1 methods	1	PhD, Mathematics, Assistant Professor
Dr. Eliot	Mathematics for teachers, 2nd in the sequence of 2	3 mathematics, proficiency test	1	PhD, Mathematics, Assistant Professor
Dr. Patrick	Combined Math & Methods, 2nd in the sequence of 2	2 combined mathematics & methods	6	PhD, Mathematics Education, Assistant Professor

Results

The full paper includes a complete episode for each teacher illustrating their use of concrete objects in a lesson focused on fractions. Here we present a summary of the cases, with additional detail about one of the cases.

A case of a new layer of abstraction: Dr. Deaver

Dr. Deaver fits a concrete representation of a problem to her textbook's definitions of fraction and of multiplication. This is an approach taken in the text, to use mathematical definitions rigorously throughout as part of an effort to convey mathematics as a coherent and connected whole. In this vignette, Dr. Deaver uses these two definitions (Beckmann, 2007):

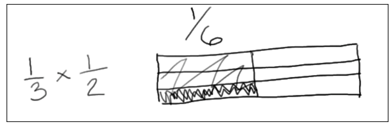
1. If A and B are whole numbers, and B is not zero, and if an object, collection, or quantity can be divided into B equal parts, then the fraction A/B of an object, collection, or quantity is the amount formed by A parts (or copies of parts) (p. 66).

2. If A and B are nonnegative numbers, then

   $$A \times B \text{ or } A \bullet B$$

   represents the total number of objects in A groups if there are B objects in each group (p. 263)

Figure 1: 1/3 •1/2 on the board in Dr. Deaver's class



Discussing the problem 1/3 • ½, Dr Deaver drew an example on the board (Figure 1) first drawing the rectangle and dividing it into two pieces vertically, then dividing it into three pieces horizontally. The following exchange ensued (Figure 2).

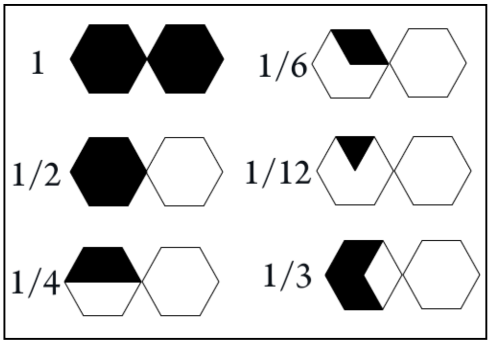
Figure 2: Dr. Deaver explains 1/3 • 1/2

1. Student 1: I still don't know why you started with 1/2, because normally in a problem you will start with, say, if 1/3 is 3, you will start with 3, to make 3 groups and make 1/2 in it.
2. Student 2: Where are you getting the whole for the ½?
3. Dr. D: 1/3 of 1/2 of the whole, so 1/2 of the whole, you are taking 1/3 of it. What you think, maybe this will help, with just whole numbers, if we just had 4 times 5, you actually, kind of start with the five, in the sense of you need four groups with 5 objects per group, so, in a sense, you started by this five objects (drew 5 circles on the board) and now you do that four times.
4. Student 3: That made a whole lot more sense than multiplication with fractions.
5. Dr. D: So you start with a half, it makes a lot of sense when you take 5 objects per group, you make 4 groups, you take 1/2 of the whole, 1/2 of the whole, you take, a third of that group, of the objects per group.

This illustrates how Dr. Deaver uses the definition of multiplication to help students make sense of fraction multiplication. In the exchange (a longer excerpt is in the full paper), the students were struggling to make the problem and the illustration fit with the definitions. The illustration itself was still an abstract entity (line 2) that seemed to this student to have come out of nowhere. The student asks, in essence, how do you know where to start? To Dr. Deaver, the idea that *any* rectangle can be “the whole” is obvious; to the student, it was another mathematical routine that he needed to learn. The very abstractness of the language (line 5) illustrates how hard it is to “concretize” the abstract idea of multiplication of fractions.

A case of concrete being used in an abstract way: Dr. Eliot

Figure 3: The 2-hexagon model



Dr. Eliot uses a 2-hexagon model with pattern blocks, and drawings of pattern blocks, throughout her teaching of fractions. Figure 3 shows the model. In this case, fully described in the complete paper, we see how a tangible object can become abstract in use, as the students struggle to connect words (rhombus, trapezoid, etc.) to numbers and ideas.

An interesting ramification of the use of this object is that Dr. Eliot explained repeatedly that the "top layer" in the physical model (Figure 4) or the "part shaded twice" in drawings of the model is the answer, a rule that turned out to have exceptions.

Figure 4: Dr. Eliot explains the "top layer" for 2/3 • 1/2

> When you actually do your models, you're going to have to do the shading on your papers because I can't look at all your models. But you're just gonna stack and the highest layer is gonna be your solution. So if you take 2/3 of a half, you with one, your two hexagons, you put half on top, right? That's your first layer. First thing you have to do is figure out what a half is. Then when you figure out what 2/3 of a half is then on top of your half you place two rhombuses. Right? So your solution would look like this, and your top layer would be the two rhombuses, and you figure out how much of your whole the top layer represents (Dr. Eliot video, 00:10:15, 2/27/08).

A case of concrete being used to concretize an abstract idea: Dr. Patrick

Dr. Patrick taught fractions using word problems and established the norm that every object had to be fully explained and linked to symbol, illustration and/or words in the problem. His method of having students go back and forth from words to pictures to numbers was a way of concretizing the meaning of fraction multiplication. In the full paper, we illustrate this through his use of the following word problem:

A batch of waffles requires ¾ of a cup of milk. You have 2 cups of milk. Exactly how many batches of waffles could you make?

Using this problem, he had students present solutions to the class, working through every number with pictures and words. In the last step of their work on the problem, he asked them to write a simple number sentence that fit the problem. His method elicited common student errors, and provided a focus on the changing whole (milk and batches).

Summary

Our analysis shows that these instructors used three different approaches to explain the abstract concepts by using concrete examples. Dr. Deaver started from the abstract idea of a fraction and stayed at an abstract level even when using real-life examples, making the examples fit into the mathematical definition. Dr. Eliot used concrete objects in a way that defined them as mathematical abstractions with meaning isomorphic to the numerical representation of fractions. Dr. Patrick used realistic word problems in a way that concretized some abstract ideas about fractions, including the idea of the changing whole. This illustrates three modes or directions for the flow of ideas in these classes: Abstract-abstract where the abstract ideas are concretized only within the abstract system itself; concrete-abstract where a concrete object is used in an abstract way; and abstract-concrete, where the abstraction is made concrete through its connection to a real object.

Striking to us was how different these classes were, all ostensibly about the same subject but offering quite different opportunities to learn mathematics. What we do not know is how these different modes of learning impact what happens when these students become teachers and actually teach mathematics. We come back to old questions of breadth v. depth; of learning mathematics in a rigorous and purely mathematical way v. learning mathematics in use; and of who has the responsibility for teaching the mathematics that falls between pure mathematics and pure method – the mathematical part of what might be called pedagogical content knowledge (Shulman, 1986), or more recently, mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008).

Significance

There is surprisingly little research on undergraduate mathematics classes for future elementary teachers: what is taught and learned, how it is taught, or how we might have greater success with this population. One reason for the lack of research is the idea that it is adequate simply to teach them the math they will teach. A list of topics extracted from tables of contents for textbooks for these courses suggests that the content is simply a review of K-8 mathematics. This study, in conjunction with other work from the larger project, shows that it is not so simple. Here, we provide insights into these classes, showing in particular how different the same content can be. We also show that mathematics classes are perhaps overlooking some of the most important mathematics these future teachers need to learn – a *meta* view of how the pieces fit together, including the mathematics of different approaches to a topic and the mathematical entailments of concrete objects used for teaching.

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