Extension from Whole Numbers to Fractions

Jungeun Park, Beste Güçler, and Raven McCrory

Michigan State University

Abstract

This paper explores how extensions of the number system from whole number to fractions are discussed in mathematics courses for prospective elementary school teachers (PSTs). First, we explain four ways of extending the number system found in the historical development of fractions: part-whole-based, measurement-based, division-based, and set-theoretical approaches. Second, we analyzed fraction lessons taught by six instructors of such courses, focusing on whether and how the instructors made extensions. Results show these instructors covered some aspects of extensions, discussing them implicitly rather than explicitly. Finally, we argue that knowledge of mathematical extensions may be beneficial to PSTs when they become teachers, for they will need to help their K-8 students understand fractions as part of a coherent number system.

Objectives

The purpose of this study is to explore mathematical ideas supporting the development of fractions from whole numbers and to investigate how this development occurs in mathematics content courses for prospective elementary school teachers (PSTs). Although PSTs may not address this extension from whole numbers to fractions with their future students, having knowledge beyond what they will teach is important to understand and help overcome struggles that children might have. It is well-known and documented that K-8 teachers and students struggle with fractions (e.g., Tirosh, Fischbein, Graeber, & Wilson, 1999; Ball, 1990; Ma, 1991). Among other problems, learners, and sometimes teachers, often apply their knowledge about whole numbers incorrectly to fractions (Erlwanger, 1973; Mack, 1990; Streefland, 1995).

To understand problems of learning and teaching fractions as close relatives of whole numbers, we explored how fractions have developed historically, and then, with the lens of history, investigated whether and how six instructors of mathematics content courses for PSTs addressed the extension of the number system when discussing fractions. In this paper, we define "fractions" as numbers in the form *a/b* where *a* and *b* are whole numbers, and *b* is not zero, and use "rational numbers" and "fractions" interchangeably. Finally, we use the term "instructors" for instructors of mathematics content courses for PSTs.

Theoretical framework

From a review of historical documents and mathematics textbooks, we see that in the history of mathematics, rational numbers were built up from whole numbers in four ways:

(a) Part-whole-based approach: finding a part of a partitioned object. Historically, this conceptualization of fractional quantity grew from ancient times when "the one" was conceived as "impartiable and indivisible" (Klein, 1968, p. 40). A fractional quantity was not considered as a number for centuries; rather, it was used as a new unit representing the part or parts of a number until Stevin (1548-1620) claimed that this quantity is a number by defining a fractional number as "a part of the parts of a whole number" (Klein, 1968, p. 290). This mirrors students' difficulties in moving beyond the part-whole concept and conceiving fractions as numbers (e.g., Erlwanger, 1973; Mack, 1990).

(b) Measurement-based approach: finding fractions from whole numbers through measurement and proportions, addressing the need for a common unit of measurement for two quantities. Historically, the term encompassing measurement and proportion is "commensurability" which was defined by the Greek mathematician, Euclid, in 300 BC as follows; "Those magnitudes are said to be commensurable which are measured by the same measure, and those incommensurable which cannot have any common measure" (Heath, 1956, p. 10). In the modern sense, if nonzero, A and B are commensurable quantities then there exists a quantity C such that $A=mC$ and $B=nC$ for non-zero whole numbers m and n. The quantity C was not considered as a number by Euclid, but as "the part or parts of a *number"* (Klein, 1968, p, 43).

(c) Division-based approach: finding the algebraic solution for an equation $ax=b$ where a and b are whole numbers and a is nonzero. This approach arises in the formal definition of a field, first conceived of by Galois in the early 19th century and formalized concretely by Dedekind in 1871 (Baumgart, 1966, p. 10). We call this a division-based approach, since the need for the fraction a/b is a result of the need to have a set of numbers where division is closed (i.e., multiplicative inverses exist and satisfy the field axioms) in order to solve algebraic problems.

(d) Set-theoretical approach: defining rational numbers as a set of ordered pairs consisting of whole numbers:

Take the set S of all ordered pairs (a,b) of integers, where $b \neq 0$. Partition the set S into subsets by the rule: two pairs (a,b) and (c,d) are in the same subset if the ratio of a to b is the same as the ratio of c to d, that is, if and only if $ad = bc$ (Childs, 1995, p. 3). This approach can be found in the 19th century and 20th century efforts to develop a rigorous foundation for mathematics; a number of mathematicians turned to arithmetic as the source for such a foundation. In the late 19th century, Cantor developed set theory, which eventually led to formal, set theoretic definitions of rational numbers. This was apparent in the "new math" movement of the 60's, building on the work of Bourbaki. An example of this exposition of rational numbers is found in an article by Brumfiel, published in one of the many reports from the School Mathematics Study Group (Brumfiel, 1966).

Methods

Fraction lessons taught by six instructors of mathematics content courses for PSTs were videotaped and complemented by field notes taken during the lessons. Researchers looked at all the videos of fraction lessons, created rough transcripts for all lessons, noted parts when the extension was discussed, and made more detailed transcripts of those parts. Each of the researchers analyzed these segments of video clips, transcripts, and field notes, and compared the results until reaching agreement.

Data sources

Data for this study came from a larger project investigating mathematics content courses taken by PSTs. This study focuses on how six of the instructors extended the number system from whole numbers to fractions and/or a need for a new number beyond whole numbers. Although most data came from first fraction lessons when they introduced fractions, some instructors explained extensions in later fraction lessons.

Results

Analyses of the lessons revealed two aspects of introducing fractions: how instructors extended from whole numbers to fractions and how they defined fractions. In this section, how one instructor, Pat, explained these extension(s) and definition(s) will be discussed in detail, and then all six cases will be summarized in Table 1. Pat was chosen because he was the only instructor who used a measurement-based approach, an effective way of teaching fractions (Lamon, 2007). Detailed analyses of other cases will be included in the final version of our paper.

Pat

In the beginning of the fraction unit, Pat explained the need for fractions using two approaches: division-based and measurement-based approaches. He started his fraction lessons with nine division word problems where a fraction is a result of division; three of them involved equal sharing, four involved comparing fractions, and the last two were measurement problems.

Figure 1. Three problems about equal sharing (Pat, April 3, 2008)

1. Four children want to share 10 chocolate bars so that everyone gets the same amount. How much chocolate can each child have?
2. Four children want to share one chocolate bar so that everyone gets the same amount. How much chocolate can each child have?
3. Four children want to share three chocolate bars so that everyone gets the same amount. How much chocolate can each child have?

Pat and his students agreed that the first three problems involve an equal sharing context. He, then, specified problems further as "equal sharing with a divisible remainder[1]," "equal sharing [of a] single unit[2]," and "equal sharing [of] multiple units[3]" in order. They compared these examples with a division problem involving an indivisible remainder.

Figure 2. Conversation about a divisible remainder and fractional thinking on April 3, 2008

Student 1: 9 fish 4 fish poles, and put the fish in there, they got two and one left over [rather than putting half a fish in two poles].
Student 2: We had another problem with cookies so that they can divide them…
Pat: The cookie one [problem] is the perfect example of equal sharing of a divisible remainder. It is a division problem that led us [to] *fractional thinking.*

In this episode, Pat connected equal sharing situations to a need for fractional thinking when the remainder, thus the dividend, was divisible quantity.

Pat also used a measurement-based approach to explain the need for fractions. Two of his division problems involved measurement.

Figure 3. Two problems about measurement (Pat, April 3, 2008)

8. It takes one can of yellow paint to paint 4 miles of road. How much yellow paint do you need to cover 10 miles of road?
9. It takes one can of yellow paint to paint 8 miles of road. How much yellow paint do you need to cover 6 miles of road?

[1] A remainder of a continuous object such as a chocolate bar
[2] The divisor is larger than 1 but the dividend is 1.
[3] The dividend is larger than 1

Pat classified these problems further depending on the answer being less than or greater than one, and related these measurement-involved situations to *fraction-based thinking* (Figure 4).

Figure 4. Pat's description about a need for fractions on April 3, 2008

> Here [Problem 8] you got a measurement division with remainder and …you have one can for four miles, so you have more than one can. Is that two and half? Is that what we would need?... So you've got measurement division with a remainder and then what you do with remainder, there is your half can. Here [Problem 9] is measurement division less than one, one can do eight miles, and for six miles you need less than one can. Also you get into the fractional quantity….It's taking this division actions we've talked about all semester and all we've done is…we put a specific context of number sizes to elicit …the concepts of *fraction-based thinking*

In both cases of equal sharing and measurement, problem statements contained only whole numbers. With the first three problems (Figure 1), Pat explained a need for "fractional thinking" in whole number division, using a division-based approach to express the amount of an object distributed into recipients. With the last two problems (Figure 2), Pat used measurement-based approach by providing a situation in which a fraction is necessary to represent a remainder less than the unit of measurement.

Pat did not explicitly explain whether "fractional quantity" is a new kind of number or how it contributes to the extension of the number system. Also, he did not give an explicit definition of fractions. Therefore, both approaches were discussed implicitly as extensions of the whole number system. He used these two approaches consistently throughout his lessons to interpret and represent fractions.

Summary

How the six instructors defined fractions and extended the number system from whole numbers to fractions is summarized in Table 1.

Table 1.

Types of Extensions and Definitions Presented by the Instructors

Instructor	Type of Extension	Definition Statement	Type of Definition
Dee	Set-theoretical approach	"a over b, where a and b are whole numbers, and b is non-zero."	Set-theoretical definition
Edie	Division-based approach	No explicit definition of fractions was given	Not applicable
Eliot	Division-based & Set-theoretical approaches	"A rational number is a number that can be written as a traditional fraction…a over b…a and b are integers b … is not zero"	Set-theoretical definition
Jamie	Division-based approach	"The denominator" as "total number of parts in a whole unit" and "the numerator" as "number of parts shaded or we count"	Part-whole definition
Pat	Division-based & Measurement-based approaches	No explicit definition of fractions was given.	Not applicable
Sam	Division-based approach	"The bottom number is how we cut this whole thing into pieces, and the total pieces, and 3 [numerator] is gonna [sic] be the number of pieces we are talking about with respect to the whole thing."	Part-whole definition

As shown in the table, some instructors presented multiple approaches to extension. Five instructors used a division-based approach. Among these, four instructors used contexts of equal sharing. Two instructors, Dee and Eliot, used a set-theoretical approach. Only one instructor, Pat, used a measurement-based approach.

The definitions of a fraction also varied across instructors. Two instructors, Dee and Eliot, used the set-theoretical definition, consistent with their extension. Two instructors, Jamie and Sam, defined fractions based on the part-whole interpretation which was inconsistent with their division-based approach to extension. Edie and Pat did not provide a definition of fractions.

The degrees of explicitness of extensions also varied among instructors. Three instructors explicitly mentioned how these approaches lead to a need for the extension of the number system. For example, in contrast to Dee, who provided the set-theoretical definition of fractions, Eliot explained why this definition implies that rational numbers include all the whole numbers. Edie and Sam also explicitly mentioned the need for a new kind of number, fraction, to represent the result of whole number division

Significance

Our analysis shows that, in contrast to the four ways fractions were developed mathematically in the course of history, only partial aspects of the development were addressed in classes for PSTs. Moreover, the instructors tended to discuss extension implicitly rather than explicitly. In most cases, approaches were limited to a division-based approach using equal sharing contexts, and the shared quantity, represented by a fraction, was often considered as a part of the whole rather than as a number.

This implicit and limited ways of discussing extension may be based on instructors' assumptions about what PSTs know (e.g., "You already know fractions, right?" Jamie, October 30, 2007). The instructors seemed to assume that PSTs already know about extension from whole numbers to fractions and other aspects of fractions. Research has reported that PSTs bring considerable knowledge about fractions to these courses although their knowledge is not always mathematically correct (Ball, 1988; Ma, 1998; Stafylidou & Vosniadou, 2004).

It seems problematic to discuss extension from whole numbers to fractions mostly using a division-based approach without explicitly mentioning that the results of division, i.e., fractions, are numbers. The contexts of sharing an object fairly among recipients are likely to position PSTs to focus on part-whole conceptions of fractions in later lessons by letting them focus on the shared piece as a part-of-the-whole. Historically, the part-whole interpretation seemed to interfere with accepting fractions as numbers for centuries. Conceiving of fractions as objects disconnected from the whole number system is also found in studies about students' understanding of fractions (Post Cramer, Lesh, Harel, & Behr, 1993; Streefland, 1995). Moreover, research has shown that teaching fraction through a measurement-based approach may be most effective in the sense that students have better sense of all other interpretations such as part-whole, ratio, operator, quotient, and relationships among them (Lamon, 2007).

Considering both the historical development of fractions and research on students' understanding of fractions, we argue that these PSTs' knowledge of fractions should move beyond the part-whole interpretation. Although other approaches are not necessarily what they will teach their K-8 students directly, having knowledge about the mathematics of the number system based on historical development of fractions may be beneficial for teaching fractions effectively. It is this kind of knowledge – more than what they will teach – that may constitute the foundation for mathematical knowledge for teaching, enabling them to understand students' difficulties and employ a variety of approaches to teaching hard-to-learn concepts. With this knowledge, they can help students understand fractions as elements of a coherent number system rather than fragmented mathematical objects subject to incomprehensible rules.

References

Ball, D. (1988). *Knowledge and reasoning in mathematical pedagogy: Examining what prospective teachers bring to teacher education.* Unpublished dissertation, Michigan State University, East Lansing.

Ball, D. L. (1990). The mathematical understandings that prospective teachers bring to teacher education. *The Elementary School Journal, 90*(4), 449-466.

Ball, W. W. R. (1960). *A short account of the history of mathematics*. New York, Dover Publications.

Childs, L. (1995). *A concrete introduction to higher algebra*. New York: Springer.

Clark, M. R., and Berenson, S.B., Cavey, L.O. (2003) A comparison of ratios and fractions and their roles as tools in proportional reasoning, *Journal of Mathematical Behavior* 22, 297–317.

Erlwanger, S. H. (1973). Benny's conception of rules and answers in IPI mathematics. *Journal of Children's Mathematical Behavior*, 1(2), 7-26. Reprinted in Carpenter, T. P., Dossey, J. A., & Koehler, J. L. (Eds.). (2004). *Classics in mathematics education research*. Reston, VA: National Council of Teachers of Mathematics.

Hannula, M. S. (2003) Locating fraction on a number line, In N A. Paterman, B.J. Doughetty, & J.T Zilliox (Eds.) *Proceedings of the 27th conference of the International Group of the Psychology of Mathematics Education (3), 17-24*. Honolulu, HI

Heath, T. L. (1956). *The thirteen books of Euclid's Elements* (2d ed.). (Vols. 1-3) New York,: Dover Publications.

Kieren, T. E. (1992). Rational and fractional numbers as mathematical and personal knowledge: Implications for curriculum and instruction. In G. Leinhardt, R. Putnam & R. A. Hattrup (Eds.), *Analysis of arithmetic for mathematics teaching* (pp. 323–372). Hillsdale, NJ: Erlbaum.

Klein, J. (1968) *Greek mathematical though and the origin of algebra.* Cambridge, Mass.,: M.I.T. press.

Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research *Handbook of Research on Mathematics Education* (pp. 629-667).

Ma, L. (1998). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum.

Mack, N. K. (1990). Learning fractions with understanding: Building on informal knowledge. *Journal for Research in Mathematics Education, 21*(1), 16-32.

Mack, N., K. (1995) Confounding whole-number and fraction concepts when building on informal knowledge *Journal for Research in Mathematics Education* 26(5), 422- 441

McCoy, N. H., & Janusz, G. J. (2001) *Introduction to Abstract Algebra.* San Diego, CA: A Harcourt Science and Technology Company

McDonald, J. N., & Weiss, N. A. (1999) *A course in Real Analysis.* San Diego, CA: A division of Harcourt Brace & Company

Post, T. R., Cramer, K. A., Lesh, R., Harel, G., & Behr, M. (1993). Curriculum Implications of research on the learning, teaching and assessing of rational number concepts. In T. P. Carpenter, E. Fennema & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 327-362). Hillsdale, NJ: Lawrence Erlbaum Associates, Publishers.

Stafylidou, S. & Vosniadou, Stella (2004) The development of students' understanding of the numerical value of fractions. *Learning and Instruction,* 14, 503-518

Streefland, L. (1993). Fractions: A realistic approach. In T. P. Carpenter, E. Fennema & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 289-325). Hillsdale, NJ: Lawrence Erlbaum Associates, Publishers.

Tall, D., (2004) Thinking through three words of mathematics, *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, (4) 281–288.

Thompson, P. W., & Saldanha, L. A. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, W. G. Martin & D. Schifter (Eds.), *A Research Companion to Principles and Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Tirosh, D., Fischbein, E., Graeber, A. O., & Wilson, J. W. (1999). Prospective elementary teachers' conceptions of rational numbers Retrieved June 11, 2004, 2004, from http://jwilson.coe.uga.edu/texts.folder/tirosh/pros.el.tchrs.html

Toluk, Z. & Middleton, J.A. 2001. The development of children's understanding of the quotient: A teaching experiment. In M. van den Heuvel-Panhuizen (ed.) *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education (4)*, 265-27