USING SEMIOTICS TO TEACH RATIONAL NUMBERS TO PROSPECTIVE ELEMENTARY SCHOOL TEACHERS

Andrea Ploucher Francis
Michigan State University
ploucher@msu.edu

Paul Woo Dong Yu
Grand Valley State University
paul@gvsu.edu

Jeremy Francis
Michigan State University
francis@law.msu.edu

Raven McCrory
Michigan State University
mccrory@msu.edu

Prospective elementary mathematics teachers should be able to understand how their future students understand number concepts. A difficult concept is that of rational numbers. Rational numbers often have complicated means of representation, signifiers and signifieds, making them difficult for students to understand and teach. In this paper, we describe how one teacher integrates a theory of semiotics when instructing prospective elementary school teachers about rational numbers. We propose that by teaching prospective teachers about semiotics, connections between signs and units are made explicit and prospective teachers will be more equipped to approach the instruction of rational numbers to future students.

Background on Obstacles to Understanding Rational Numbers

Lamon (2007) suggests that fractions are a subset of rational numbers, in that fractions are notational and are "non-negative rational numbers" (p. 635). We are interested in how to instruct preservice elementary school teachers to move beyond simply the use of symbols (notational systems in which there are two integers written with a bar between them) to a conceptual and transferable understanding of what those symbols represent.

Preservice elementary teachers often have trouble understanding and ultimately teaching future students about rational numbers (Graeber, Tirosh, & Glover, 1989; Harel et al., 1994; Simon & Blume, 1994). One key issue that prospective teachers face when they are trying to understand and work with rational numbers is whether a fraction is related to division, is a type of multiplication, or is a ratio of some sort (Ni, 2001). For example, Behr and his colleagues (Behr et al., 1993, 1994) demonstrated that transformation involved in solving problems with rational numbers use compositions and recompositions of units. The part–whole construct for 2/3 suggests two interpretations: two-third as parts of a whole are two one-third units, i.e., 2(1/3-unit)s, or two-thirds as a composite part of a whole is one two thirds unit, i.e., 1(2/3-unit). In the number sentence $2 \div 3 = 2/3$, fractions are related to the idea of division. However, in another sense, in order to get 2/3 you must first define 1/3 and then multiply it by the number of 1/3rds that you have. Further, if the numerator and denominator are meant to express a ratio—like there are two dogs for every three cats, then 2/3 cannot be thought of in either of the above ways. You cannot have 2/3 of a dog.

Purpose of this Study

In this paper we consider how semiotics instruction can be used to help preservice elementary school teachers learn about rational numbers. While there may be many reasons why rational numbers are hard for students to understand, from a lack of prerequisite knowledge to a lack of working memory to hold multiple numbers in mind at one time, we are going to focus on how different meanings denoted by a single representation of a fraction in different relevant contexts may lead to an inability to fully understand rational numbers. We describe how one professor used a dyadic, or two part, semiotic framework developed by Ferdinand de Saussure (1957) to help instruct preservice elementary school teachers about rational numbers. Furthermore, by using this framework, the teacher was able both to decompose the problems themselves and to give the preservice teachers a way to understand how their future students will view the problems.

Theoretical Framework: Why the Study of Rational Numbers is a Semiotic issue

Semiotics, broadly conceived, "is concerned with everything that can be taken as a sign" (Eco, 1976, p. 7), including "images, gestures, musical sounds, objects…these constitute, if not *languages*, at least systems of signification" (Barthes, 1967, p. 9). Ferdinand de Saussure (1957) described a linguistic sign as a two-sided entity made up of a signifier and signified. A signifier is the material aspect of the sign whereas the signified is the mental concept associated with the material symbol. For example, the English word "tree" is made up of the material sounds /t/, /r/, and /e/ (the signifier) as well as the mental concept we each hold of what it means to be a "tree" (the signified).

In addition to the simple model of a signifier and signified, one must also bear in mind the community in which this relationship takes place. Saussure reminds us that regardless of the signifier a linguistic system uses "to designate the concept 'tree,' it is clear that only the associations sanctioned by that language appear to us to conform to reality" (1957, p. 66-67). Thus, there are many different signifiers that can represent the same signified. In the field of semiotics, "langue" refers to the collection of signs, the overall system of signification, that permit individual speech utterances. Put another way, "langue" is "language minus speech," the structure that permits individual utterances (Barthes, p. 14). Depending on how the words are combined in a phrase or what words are chosen in a particular phrase, the concept of the signified may change. In other words, the meaning of a sign is not contained within the signifier or signified alone, but develops within a phrase and within a community as well. To make sense of the sign, not only do the signifer and signified need to be taken into account, but the context that contains that sign must also be considered. One type of sign may be signified by a written symbol. Therefore, it follows that one way to study and begin to understand rational numbers is by using semiotics.

Taking the idea of rational numbers as signs, we can see how it can be difficult for students to make sense of them. A single signifier of a rational number, like two whole numbers with a bar in between them (ex – ½), can take on multiple meanings. Consider the following uses of rational numbers taken from a popular book used to teacher prospective elementary teachers mathematics (Billstein, Libeskind, Lott, 2007, p. 299):

Table 5-1 Uses of Rational Numbers

Use	Example
Division problem or solution to a multiplication problem	The solution to 2x = 3 is 3/2
Partition, or part, of a whole	Joe received ½ of Mary's salary each month for alimony.
Ratio	The ratio of Republicans to Democrats in the Senate is three to five

Probability	When you toss a fair coin, the probability of getting heads is ½.

In the example, a rational number could be used as division, to partition something, as a ratio, or even as a probability. Similarly, Kieren's semantic analysis of rational number identifies five "subconstructs" of rational numbers. They are part–whole, ratio, quotient, operator, and measure (Behr et al., 1992; Kieren, 1976). Depending on the context, the meaning of the rational number – even if it is represented by the same numbers – changes.

Consider the following question out of context. What does 2/3 mean? This signfier, 2/3, could mean any of the following (the list is certainly not exhaustive):

- 2 candy bars shared equally by 3 people, each person gets 2/3
- 2 dogs for every 3 people
- 2 ÷ 3 = 2/3
- 1 ÷ 3 = 1/3 (partitioning), there are 2 1/3rds = 2/3 (iterating)
- 2 parts, with 3 equal parts to make a whole
- 2/3 = 4/6 = 6/9, etc.
- It could be 2/3 of a number greater than one, like 6, which is equal to 4
- It could be 2/3 of a number less than one, like 1/6, which is equal to 1/9

We posit that one difficulty students have when trying to first understand and then later teach rational numbers is that they must navigate across multiple meanings, or signifieds, for the same symbol, or signfier, a relatively common task in spoken and written language, but not necessarily common in the understanding of numbers. We use the phrase 'semiotic dissonance' to describe the difficulty or inability for a person to meaningfully construct a sign from a signified (meaning) and its associated signifier (symbol). Further, when a teacher develops the ability to navigate across meanings, he or she must be able to step back and understand how future students must then navigate across meanings. This understanding is the basic semiotic framework that we will refer to for the rest of this paper.

Mode of Inquiry: A Case of One Teacher Using a Semiotic Framework to Teach Rational Numbers to Prospective Elementary Teachers

As part of a larger study that explores the mathematics content taught to undergraduate prospective elementary teachers, this paper focuses on one of seven instructors who were video-taped while they taught fraction lessons. The video data was supplemented by the field notes taken during instruction and of an interview of the instructor. For more information about the project, including student test data and teacher surveys, see McCrory (2008). Since in this paper we are concerned with overcoming the difficulties students face when trying to work with rational numbers, we looked at one college professor in more detail. For this project, over 15 hours of video data was gathered for this instructor. The analysis of the video tapes used an Iterative Refinement Cycle (Lesh & Lehrer, 2000) model in which multiple interpretive cycles were used on the data. The first interpretive cycle was used to identify those issues that pertained to general pedagogy and classroom culture. The second cycle was used to identify those issues that reflected the semiotic framework that framed this research investigation. The third cycle was used to establish explicit connections between the instructor's pedagogical decisions in order to make clear to the students the semiotic issues in their problem solving process. Finally, throughout the entire iterative viewing and interpretive process, the analysis of the other data source, including field notes and the exit interview with the instructor, was used to help inform the context and nature of the classroom discourse.

Below we provide excerpts of the interpretive narrative and discuss how Pat (a pseudonym), a professor of prospective elementary school teachers, used semiotics to instruct his students about rational numbers. The following episodes from Pat's class show how an understanding of semiotics can help preservice elementary teachers understand rational numbers and ultimately how to understand how to navigate these teachers' future students' misconceptions of rational numbers.

Results

Before delving into the difficult concept of rational numbers, Pat did a short presentation introducing the students to semiotics. In the presentation, Pat first defined a linguistic sign as a "…two sided entity, a dyad, between the signifier (symbol) and the signified (meaning)." He then gave the following visual example of linguistic sign using the concept of 'dog':

sign → <u>Spoken word "Dog"</u>
Concept of Dog

Pat would refer back to this simplified semiotic framework when semiotic dissonance arose in problems that had the same numbers, but different units, to explicitly show students how the semiotic framework provided insight into their confusion. Pat specifically chose the waffle and cookie problems below to include two meanings of division: partitive/sharing (how many/much per group?) and quotative/measurement (how many groups?).

> The Waffle Problem: *A batch of waffles requires ¾ of a cup of milk. You have two cups of milk. Exactly how many batches of waffles could you make?*
>
> The Cookie Problem: "*You have 2 cups of flour to makes some cookies. This is ¾ of what you need for one full recipe. How many cups of flour are needed for a full recipe?* (Class handout, 4/10/08)

In his class, Pat began by having students work in six groups of four students to solve the above problems. When Pat asked for solutions to the first problem, four of the six small groups each gave a different answer (2, 2 ¼, 2 2/3, 2 3/8), only one of which was actually correct. Clearly this question posed a series of problems for this class. Pat proceeded to have the group that had 2 ¼ go to the board and write how they solved the problem for the class. One student said, "I knew that 1 cup of milk was four fourths," and then wrote out on the white board 4/4 + 4/4 = 8/8 to represent the 2 cups of milk. He continued saying, "I know ¾ a cup is batch so I took away 6/8 for two batches," while writing 4/4 + 4/4 = 8/8 – 6/8 = 2/8. The student concluded by saying that he had 2 batches so far, represented by the 6/8, and a ¼ leftover, simplified from the 2/8, giving a final answer of 2 ¼ batches. Another student quickly pointed out that 4/4 plus 4/4 was actually 8/4, not 8/8. Using that fact, Pat reworked the problem on the white board writing 4/4 + 4/4 = 8/4, and 8/4 – 6/4 (for the two batches of waffles) = 2/4 = ½. The class then began to discuss the meaning of the ½, whether it meant ½ a batch or ½ a cup. After some class discussion on how to interpret the ½, a third student pointed out that the ½ left over was not ½ a BATCH, but rather was ½ a CUP and that ½ cup was the same as 2/3 of a batch. To illustrate his point, the student drew a pictorial representation of the problem (figure 1) trying to show how ½ cups of milk was equivalent to 2/3 batch of waffles.

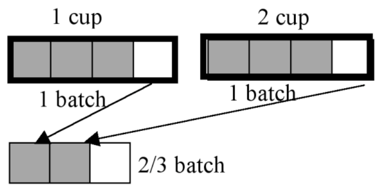

Figure 1 – Pictorial solution to the waffle problem

After the students explained why they got their answers, Pat brought the class back and explained how both a cup and a batch could be a whole. Pat stressed that the students needed to attend to the context, and said, "technically the ½ is not wrong until you put a name to it. You say ½ a batch. That's not true, because it's half a cup." (from video on 4/10/08 and 4/15/08).

This is clearly a semiotic problem, where the signifier, or visual mark "1/2," had taken on two different meanings, depending on the signified, or concept, with which the students associated the signifier to construct their sign. One student incorrectly associated the ½ to 'batches' while another student correctly associated the ½ to 'cups,' which was equivalent to 2/3 of a batch. In this case, the single signifier, ½, could only be meaningfully associated with one meaning, cup. We describe this phenomenon with the phrase 'semiotic mismatch,' in which a signifier is incorrectly associated with a particular signified as determined by the context. As illustrated in this excerpt, by explicitly showing students how signs can mean different things depending on the context, the students gained insight in how to determine what the accurate signified, or intended meaning, was in the problem.

When Pat moved to a discussion of the cookie problem, he encouraged the students to use a semiotic framework not only when approaching their own understanding, but also in future instruction of rational numbers to elementary school students as well. As with the waffle problem, Pat first had the class work in small groups to come up with a way to model their solution. Pat had two groups put up two solutions. The first solution, which was an algebraic solution, looked like this:

2 cups = 8/4 cups
2 divided by ¾
8/4 divided by ¾
8/4 x 4/3 = 8/3
2 2/3 cups

The second solution (figure 2), copied in the notes as replicating the board drawing, was a pictorial strategy.

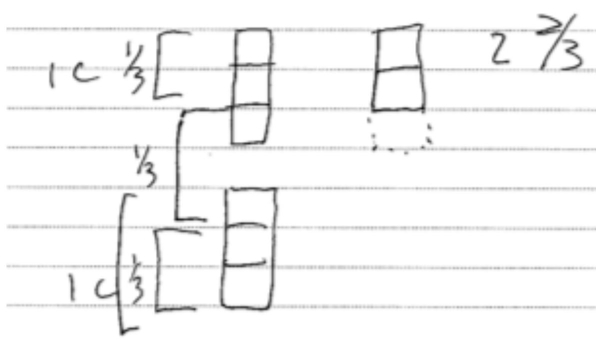

Figure 2 – Pictorial solution to the cookie problem

At this point, Pat gave the students some time to think about how they would explain each of the above answers to their future students. Pat discussed how this was a content pedagogy course, so he wanted them to understand the content, but then, as future teachers to be able to explain the mathematics with clarity. While no one seemed eager to attempt to explain the first solution strategy, one student did label and explain the second strategy. First, she labeled the model (figure 3) and noted that she thought it was important to label cups on one side and ¼ batch on the other side so students would not get confused. Here the student had moved from having semiotic dissonance to realizing the importance of being consistent with the use of symbols when there are multiple signifieds, quantities of cups and recipes, as a result of the context.

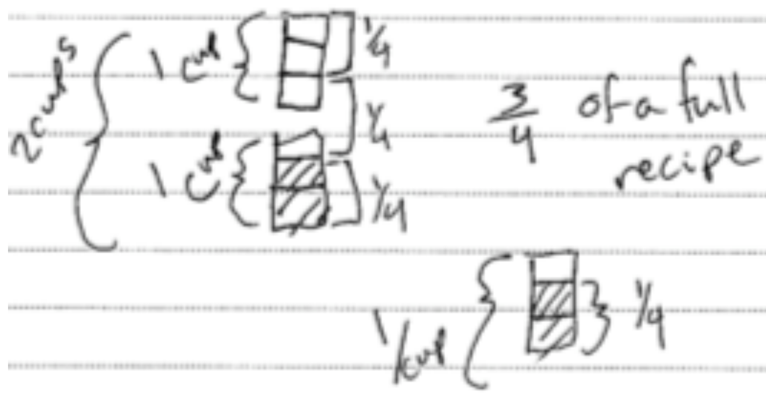

Figure 3 – Pictorial solution to the cookie problem with student explanation

Pat waited for the students to comment as the model was explained. He then pointed out that the two shaded boxes in the drawing, 1/3 of 2 cups and ¼ of a recipe were also both equal to 2/3 cups. In this case, the signifier represented by the two shaded boxes, could be associated with three different meanings, the signified, depending on context.

After students shared solutions of the cookie problem, Pat asked the groups to think about what mathematics was necessary for kids to solve the problem using the different strategies presented. The class decided that to solve the problem using strategy 1, the algebraic solution, the child would need to know improper fractions, the division of fractions, fraction algorithms, and whole number operation facts. On the other hand, for strategies 2 and 3, the pictoral strategy, the child would need to know how to partition pictures, split quantities up, and recognize the concept of a changing whole. One of the students specifically pointed out that in strategy 1, number facts were necessary while in strategies 2 and 3 a conceptual understanding of division was needed. Furthermore, looking back over the three representations, one could see that while the first strategy was mathematically correct, a student would not necessarily need a conceptual understanding to solve the problem. There were no units labeled, and thus semiotic confusion could arise in children. In the second solution, the diagram was not labeled adequately, which could also lead to semiotic confusion. The third solution had taken into account much of the semiotic framework that had been taught, including labeled pieces so that a child could more easily figure out what is signified by the images and numbers.

By analyzing the three representations of the cookie problem and determining what langue, or semiotic context, was needed to solve the problem using the two strategies, Pat enabled his students to begin to think about how they could instruct future students about rational numbers. He emphasized the importance of labeling the units and how the changing whole could be tricky for their future students to understanding and learn. He presented them with a semiotic framework to be able to create meaningful signs by connecting the appropriate signifiers with the appropriate signifieds, depending on the knowledge available to his students as well as his students' future school aged students.

After discussing the waffle and cookie problems individually, Pat asked the groups to determine the difference between the two problems. While both problems used the same numbers (2 and ¾), the same operation (division), and had the same answers (2 2/3), the contextual difference led to different solution strategies and conceptions of the fractional quantities. Again, Pat wanted the students to understand that the langue of the problem affects the ultimate meaning of the signifieds (amount) and of the signifiers (visual or auditory number itself). By now, the students had decomposed both problems and quickly answered that: 1) In the problems they are asking two different questions; 2) In the first you have everything you need. In the second problem you need to figure out what extra represents. You don't have everything you need. 3) In the first problem, ¾ makes a whole, in the second it is ¾ of what is a whole; and 4) In the first problem, the measurement is division, where you know the amount of groups and want to know what the whole is and the second is partitive division, in which you know how many groups/parts are in the whole. By analyzing the problems and giving students a semiotic framework to allow them to determine the meanings of the symbols being used, the students were able not only to understand the rational numbers themselves, but were also able to conceptualize what kinds of mathematical knowledge their future students would need to understand such problems.

Discussion

As noted in the introduction, preservice elementary teachers often have trouble understanding how to conceptualize and use rational numbers. Without adequate conceptual understanding, this lack of content knowledge will be perpetuated when these students go to teach their future elementary school students about rational numbers. This paper has shown how one teacher has used a framework of semiotics to help students understand, use, and ultimately instruct each other about rational numbers. Whether this knowledge transfers to the elementary classroom is an area that needs further study. Nonetheless, the preservice teachers in this case study clearly showed growth in their understanding of rational numbers through the use of a semiotic framework. The study of semiotics helps students to interrogate "how" things mean, not just what they mean, that a secondary system underpins the superficial representations of concepts with which they have become so familiar. Further, breaking down the problems using semiotics makes explicit what we often think we are doing implicitly. Finally, this explicit decomposing of problems allows students to begin to see where their future students may encounter difficulties when trying to understand rational numbers.

Acknowledgement

This research is funded by the National Science Foundation (Grant No. 0447611). The authors wish to thank the instructor who generously participated in this project and the other team members – Rachel Ayieko, ChangHui Zhang, Beste Gucler, Rae-Young Kim, Jane-Jane Lo, Jessica Liu, Jungeun Park, and Helen Siedel – who collected data and participated in discussions that made our analysis possible.

References

Barthes, R. (1967). *Elements of semiology*. New York: Hill and Wang.

Behr, M. J., Harel, G., Post, T. R., & Lesh, R. (1992). Rational number, ratio, and proportion. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 296–333). New York: Macmillan.

Behr, M., Harel, G., Post, T., & Lesh, R. (1994). Units of quantity: A conceptual basis common to additive and multiplicative structures. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 123-180). Albany, NY: SUNY Press.

Behr, M., Harel, G., Post, T., & Lesh, R. (1993). Rational Numbers: Toward a Semantic Analysis - Emphasis on the Operator Construct. In T. Carpenter, E. Fennema & T. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 13-47). Hillsdale, NJ: Lawrence Erlbaum Associates.

Billstein, R., Libeskind, S., & Lott, J. W. (2007). *A problem solving approach to mathematics for elementary school teachers* (9th ed.). Boston: Pearson Addison Wesley.

Eco, Umberto. (1976): *A Theory of Semiotics.* Bloomington, IN: Indiana University Press/London: Macmillan.

Graeber, A., Tirosh, D., & Glover, R. (1989). Preservice teachers' misconceptions in sovivng verbal problems in multiplication and division. *Journal for Research in Mathematics Education, 20(1),* 95-102.

Harel, G, Behr, M., Post, L., & Lesh, R. (1994). The impact of number type on the s solution of multiplication and division problems. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 363-384). New York: State University of New York Press.

Kieren, T. E. (1976). On the mathematical, cognitive, and instructional foundations of rational numbers. In R. Lesh (Ed.), *Number and measurement* (pp. 101–150). Columbus, OH: Eric/SMEAC.

Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In *Handbook of Research on Mathematics Education* (pp. 629-667).

Larson-Novillis. C. (1979). Locating proper fractions on number lines: Effects of length and equivalence. *School Science and Mathematics, 53,* 423–428.

Lesh, R., & Lehrer, R. (2000). Iterative refinement cycles for videotape analysis of conceptual change. In A. E. Kelly & R. A. Lesh (Eds.), Handbook of research design in mathematics and science education (pp. 992). Mahwah, NJ: Lawrence Erlbaum Associates.

McCrory, R. (2008). What matters: Mathematics classes for elementary teachers. PDF of PowerPoint slides presented at the University of Georgia Mathematics Education Colloquium, December 4, 2008. Retrieved from http://meet.educ.msu.edu/pubs.htm.

Ni, Y. (2001). Semantic domains of rational Numbers and the acquisition of fraction equivalence. *Contemporary Educational Psychology 26,* 400–417

Saussure, F. (1957). Course in general linguistics (Baskin, W. Trans.). New York: Philosophical Library.

Simon, M.A., & Blume, G.W. (1994). Building and understanding multiplicative relationships: A study of prospective elementary teachers. *Journal for Research in Mathematics Education, 25(5)*, 472-494.