AFFORDANCES OF VISUAL REPRESENTATIONS: THE CASE OF FRACTION MULTIPLICATION

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This study focuses on two instructors who taught a mathematics course designed for prospective elementary teachers and explores which interpretations of fractions they addressed and how they used visual representations when discussing fraction multiplication. Our findings indicate that the distinct interpretations of rational numbers can turn out to be quite intertwined during actual practice. As a result, it might be challenging to extract meaning from the visual representations, especially when the problems are not situated in a context, unless instructors explicitly attend to the interpretations underlying those representations.

Introduction

It is challenging to blend research on fractions with classroom teaching for several reasons. First, the development of fractions in the classroom is complex and non-linear. Second, "teachers are not prepared to teach content other than part-whole fractions" (Lamon, 2007, p. 632) and thus we may not see in reality the ideas and constructs that research suggests in theory. Third, teachers may not explain details supporting their choices of problems and representations, making them invisible to research. In this study, we explore some of the complexities of multiplication of fractions in the context of preservice teacher education.

There is evidence that part-whole has been the most dominant realization of fractions for students as well as preservice and inservice teachers (Domoney, 2002; Sowder, Philipp, Armstrong, & Schappelle, 1998; Tirosh, Fischbein, Graeber, & Wilson, 1999). Besides part-whole, Kieren (1980) proposed four other subconstructs or interpretations for fractions: measure, operator, quotient and ratio. Each of these interpretations may be illustrated with multiple representations, including numbers, and various discrete, linear, and area models and more. In her longitudinal study of six classes from grades 3 to 6, Lamon (2007) noted that the students who were exposed to these five subconstructs developed deeper understanding of rational numbers and proportional reasoning compared to students in the control group who received traditional instruction that did not explicitly attempt to use multiple interpretations of rational numbers. (Note that, in this paper we use "fraction" and "rational number" interchangeably, acknowledging that there are important and contested mathematical differences between the two.) As a result, she considers being able to "move flexibly between interpretations and representations" as one of the key elements of understanding fractions (Lamon, 2007, p. 636). Therefore, understanding of fractions entails experience with multiple interpretations (Kieren, 1980) as well as experience with multiple representations of fractions.

According to Izsák (2008), research on teachers' knowledge of fraction multiplication is not as extensive as the research on fraction division and decimal multiplication. There is a body of research indicating that teachers find it difficult to construct appropriate representations for fraction multiplication (Armstrong & Bezuk, 1995; Sowder et al., 1998; Tirosh et al., 1999). In this paper, we look at two instructors who teach mathematics content for undergraduate prospective elementary school teachers. The study addresses the following question: Which interpretations of fractions do instructors teaching elementary mathematics content to undergraduate preservice teachers concentrate on as they teach fraction multiplication, and how do they represent these ideas to students?

Theoretical Framework

The *part-whole* realization of rational numbers entails "the partitioning of a continuous quantity or a set of discrete objects into equal-sized parts…" (Sowder et al., 1998, p. 8). Therefore, this subconstruct requires understanding of the whole and the ways in which it may be partitioned. The realization of a rational number as a *measure* "occurs when we want to measure something but the unit of measure does not fit some whole number of times in the quantity to be measured" and so "demands that the rational number be understood as a number, as a quantity, as how *much* of something" (Sowder et al., 1998, pp. 9-10, italics in original). A rational number acts as an *operator* when it is interpreted as

> a function that can operate on a continuous region as a stretcher or shrinker or on a set as a multiplier or divider, in either case serving as a function machine that operates on one value to form an output of another value. (Sowder et al., 1998, p. 11)

A rational number can also be realized as a *quotient*. "A fraction $^{a}/_{b}$ can also represent the quotient $a \div b$; that is, a and b are integers satisfying the equation $a=bx$" (Sowder et al., 1998, p. 11). Finally, when we realize a rational number $^{a}/_{b}$ by means of the comparative relationship between a and b, we are thinking of the rational number as a *ratio*.

Among the five subconstructs of rational numbers, the operator subconstruct seems to be the most effective for fraction multiplication (Behr, Harel, Post, & Lesh, 1993; Izsák, 2008; Sowder et al., 1998). Lamon's (2007) findings also suggest using the measure subconstruct for fraction multiplication since it might enable the extension of the operator subconstruct. She exemplifies the measure and the operator subconstructs and their meaning for the fraction $^{3}/_{4}$ as follows:

Interpretations of 3/4	**Meaning**
Measure "$3(\frac{1}{4}$-units)"	$\frac{3}{4}$ means a distance of $3(\frac{1}{4}$-units) from 0 on the number line or $3(\frac{1}{4}$-units) of a given area.
Operator "$\frac{3}{4}$ of something"	$\frac{3}{4}$ is a rule that tells how to operate on a unit (or on the result of a previous operation): multiply by 3 and divide the result by 4 or divide by 4 and multiply the result by 3. This results in multiple meanings for $\frac{3}{4}$: $3(\frac{1}{4}$-units), $1(\frac{3}{4}$-unit), and $\frac{1}{4}$ (3-unit).

Figure 1: Portion of the table Lamon (2007, p. 654) uses when she addresses alternative instruction strategies to the part-whole interpretation of fractions.

In this paper, we investigate how instructors of preservice elementary teachers utilize these interpretations as they represent fraction multiplication.

Methods

Data for this study comes from a larger project that explores the mathematics content taught to undergraduate prospective elementary teachers. This paper uses data from two instructors, collected through observations of their classes when they taught fractions. Particular attention was given to the visual representations instructors used when addressing fraction multiplication. Fraction lessons were videotaped and portions of the tapes where the instructors discussed fraction multiplication were transcribed. Field notes taken during instruction supplemented the data.

We report on two instructors for whom we will use the pseudonyms Eliot and Sam. These instructors form contrasting cases with respect to the number of visual representations they used when discussing fraction multiplication. Eliot primarily relied on a single visual representation across the problems she solved whereas Sam used multiple representations for each of the problems she worked on. Given this, we investigated the possible impact of this difference on the interpretations of fractions the instructors facilitated in their classrooms. We used snapshots from the classroom videotapes for Eliot's representations to keep the authenticity of her representations on the whiteboard in her classroom. Sam's video snapshots were not clearly visible since she used a blackboard so we used field notes to reproduce her drawings. The researchers checked the fidelity of the field notes with Sam's actual representations in the video clips. We initially analyzed the data with respect to the subconstructs underlying the visual representations the instructors used for each problem individually and then compared our results until we reached consensus. In this respect, we used a form of competitive argumentation (VanLehn & Brown, 1982) during our data analysis.

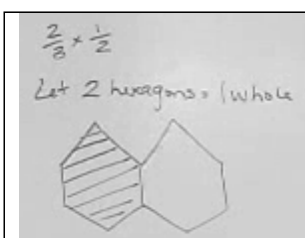
Results

Eliot's representations of fraction multiplication

Eliot based her initial discussion of fraction multiplication on whole number multiplication. Although her initial introduction encouraged students to think about fraction multiplication in terms of repeated addition, Eliot also mentioned that multiplication does not necessarily lead to a larger number in the case of fractions. After these, she explicitly pointed out the word *of* means *to multiply* and started modeling fraction multiplication problems using diagrams. Throughout her discussion of multiplication of fractions, Eliot consistently used a visual diagram in which two hexagons were considered as one whole. At the very beginning of her fraction lessons, Eliot introduced these hexagons and their subunits consisting of triangles, rhombi (Eliot used the word rhombuses instead of rhombi so we will stick with her word use), and trapezoids using pattern blocks. Afterwards, she kept on using the same idea by drawing the hexagons on the board. Below is the relationship among the units and the subunits:

2 hexagons (the whole) = 12 triangles = 6 rhombuses = 4 trapezoids
1 rhombus = 2 triangles
1 trapezoid = 3 triangles

Eliot assumed her students knew how to compute fraction multiplication and briefly mentioned the rule. She focused only on modeling during her discussion of multiplication of fractions. For all the problems she worked on, except for one, she used the diagram in which two hexagons referred to one whole. For example, she modeled $\frac{2}{3}\times\frac{1}{2}$ as follows:

	Eliot first modeled $\frac{2}{3}\times\frac{1}{2}$ by considering it as $\frac{2}{3}$ of $\frac{1}{2}$. She shaded one hexagon noting that it was the half of her whole, which was two hexagons.

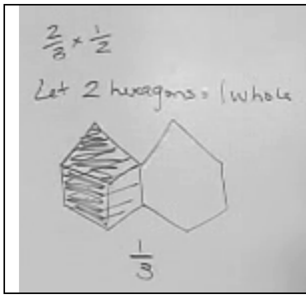
	Eliot: "What is two thirds of my half? I need to split my half into three pieces and shade two again" (The class was familiar with representing thirds by rhombuses). Therefore, Eliot split the hexagon into three rhombuses and shaded two of them one by one. She then asked what the double shaded region was in terms of the whole. Two rhombuses made up a third of the whole, so Eliot wrote $1/3$ as the answer.

Figure 2: Eliot's representation of $2/3$ of $1/2$

Eliot's wording while shading $2/3$ of one hexagon might be considered as recursive partitioning when she split the hexagon into three equal pieces and shaded two of them. Given this, she might be using the idea of finding part of a part by applying the notion of part-whole recursively. On the other hand, it is also possible that Eliot operated on the hexagon that represented $1/2$ as she split it into three parts (divide by 3) and then shaded two of them (multiply by 2). In this respect, given the description in Figure 1, she might have also used the operator subconstruct. Therefore, for this problem, it is not explicit which subconstruct she is particularly attending to. Eliot then modeled $2/3 \times 1/2$ again, this time considering it as $1/2$ of $2/3$:

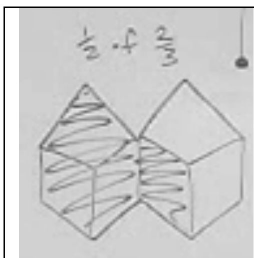
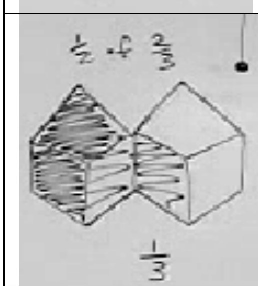
	Eliot noted that in order to take the half of $2/3$, we first needed to know what two thirds of the whole was. She asked, "two-thirds is how many rhombuses? Four" (Eliot used the equivalent fraction $4/6$ for $2/3$ to be able to represent it with rhombuses. The students were familiar with this use). She then divided the hexagons into six rhombuses and shaded four of them.
	Eliot: "What is half of my four rhombuses? Two rhombuses. How much of my whole is shaded twice?" (Note that she used the names of the geometric shapes rather than saying 'what is a half of two thirds'?) She then shaded two rhombuses out of the four rhombuses she already shaded and wrote $1/3$ as the answer.

Figure 3: Eliot's representation of $1/2$ of $2/3$

When finding the half of four rhombuses, Eliot considered four rhombuses as another whole. It is likely that she used part-whole interpretation with recursive partitioning here since she made the number $2/3$ concrete by naming it "four rhombuses". Then half of four rhombuses would be equal to two rhombuses. On the other hand, she often emphasized in her previous classes on fractions that students needed to think about this model in terms of area. For example, when modeling addition of fractions with these hexagons, she said, "we are merging the areas together to find out how much of our same whole the new area takes up". Similarly, when she explained why $3/4$ of two hexagons would be equal to three trapezoids, she mentioned, "because we can cover three fourths of the area of our whole using three trapezoids". Given this, she might also be attending to the measure subconstruct (See Figure 1) as she modeled $1/2$ of $2/3$.

Eliot modeled $\frac{3}{4}\times\frac{1}{3}$ by considering it as $\frac{1}{3}$ of $\frac{3}{4}$:

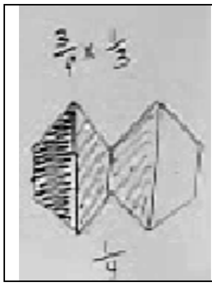
	Eliot mentioned that three fourths of the whole would be three trapezoids and shaded them in her drawing. She then asked, “what is a third of three trapezoids? One trapezoid. *Because when you have three of anything, a third of it is one of them*”. She then shaded one trapezoid out of the already shaded three trapezoids and asked how much of the whole was shaded twice. Students answered by saying a fourth. Eliot wrote $\frac{1}{4}$ as the answer.

Figure 4: Eliot’s representation of $\frac{1}{3}$ of $\frac{3}{4}$

Note that Eliot talked about one third of $\frac{3}{4}$ as one third of three things (trapezoids), which would be the recursive application of the part-whole subconstruct. Here, it seems relatively clear that Eliot used recursive partitioning for finding the part of a part rather than attending to the operator subconstruct. However, when the numerator of the operator is equal to 1, it might also be difficult to identify from the visual representation whether the operator subconstruct or the part-whole interpretation is used.

Eliot might have used the operator subconstruct when she modeled $1\frac{1}{4}\times\frac{2}{3}$ considering it as $\frac{2}{3}$ of $1\frac{1}{4}$. The following is the picture she drew:

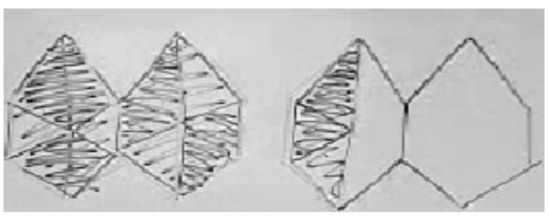


Figure 4: Eliot’s representation of $\frac{2}{3}$ of $1\frac{1}{4}$

Eliot noted that $1\frac{1}{4}$ would be equal to a whole plus an additional fourth. Since a pair of hexagons corresponded to one whole, she drew two pairs of hexagons. She shaded all of the first pair and a trapezoid in the second pair that corresponded to the additional fourth. She then said, “You want to ask yourself what is two thirds of the shaded portion? I need to think of a way to cut that into three equal portions…I guess I am going to cut everything into triangles”. After this, she split the shaded portion, which is $1\frac{1}{4}$, into triangular portions. She then noted she was going to shade “two out of every three”. Given Eliot’s previous examples and arguments, one can again assume that she used part-whole interpretation with recursive partitioning. On the other hand, it is relatively clear in this example that she operated on the fourths in $1\frac{1}{4}$ since she split every fourth in $1+\frac{1}{4}$ into three equal portions (divide by 3) and then took “two out of every three” *fourths* (multiply by 2), which would suggest she was attending to the operator subconstruct. Eliot’s consideration of $1\frac{1}{4}$ as $1+\frac{1}{4}$ when splitting each fourth into its thirds and taking two out of three fourths also signaled an implicit use of the distributive property.

Representations of fraction multiplication in Sam’s class

Sam also started her discussion of fraction multiplication with whole number multiplication. She structured her class around the three cases: a whole number multiplied by a fraction, a fraction multiplied by a whole number, and a fraction multiplied by another fraction. Similar to Eliot, Sam also mentioned the relationship between the word *of* and multiplication. Sam did not discuss mixed number or improper fraction multiplication. Unlike Eliot who used a single visual representation for each problem, Sam used a variety of visual representations (some of which were initiated by her students) for the problems she worked on. Her representations consisted of pie diagrams, rectangular area models and also the number line. For example, while modeling $\frac{1}{2}\times\frac{1}{3}$, Sam considered it as $\frac{1}{2}$ of $\frac{1}{3}$ and used the following visuals:

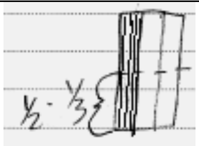
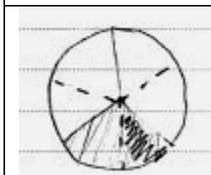
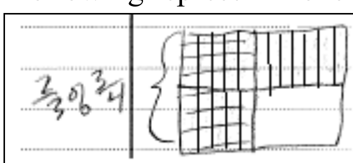
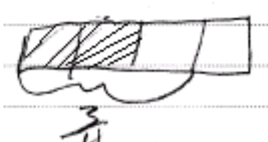
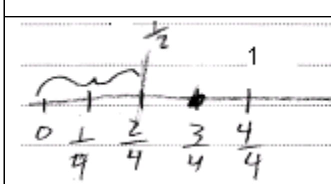
	Sam drew this model mentioning, "we draw a whole and then have one third. We then think about what one half of that third is". She split the third she shaded into two parts horizontally following a student's suggestion. She then labeled the portion that corresponded to $\frac{1}{2}$ of $\frac{1}{3}$.
	Sam then asked students if she could split the shaded third into two parts vertically by saying "I will split this to halves." She drew dotted lines and shaded one strip as shown . The students agreed that she could do this vertically.
	Sam: "How about trying other models? How about a pie?" After saying this, she drew a pie diagram and divided it as shown with solid lines. She then added the dotted lines and said "because of that, we have to cut the other parts, so this one half of one third will be what?" A student answered "one sixth of the whole". Sam confirmed this was the correct answer.

Figure 5: Sam's representations of $\frac{1}{2}$ of $\frac{1}{3}$

Sam seemed to be using the part-whole subconstruct with recursive partitioning for these representations. However, it is also possible that she might be using the operator subconstruct when she the split the third into two parts and shaded one part. On the other hand, Sam went on and split the other thirds into halves after this step for each of the visuals. In this respect, it is more plausible that she used the part-whole subconstruct with recursive partitioning rather than the operator. In general, it is difficult to distinguish the operator subconstruct from the part-whole subconstruct in situations where the numerator (of the operator) is 1 or when the denominator is equal to the numerator of the operand. The latter is shown when Sam elicited the following representations for $\frac{2}{3}\times\frac{3}{4}$:

	A student drew this representation first by drawing the rectangular region as the whole, then cross-partitioning it and shading three fourths. The student then said, "I will take two thirds of this three fourths" and shaded the two parts as shown on the right. Sam did not add any other explanation for this picture and asked what other models the students could use.
	Another student drew this picture. In Sam's class, this model is referred to as the bar diagram and is mostly used for situations involving measurement. The student first split the bar into four parts and then labeled three fourths of the whole. She then shaded two parts one part at a time. Again, Sam did not have any additional comments about the picture and asked the class if they could model the same problem using the number line.
	Sam drew this model herself. She first put the numbers 0 and 1 on the line and then divided the interval into four parts. She marked three fourths. She then labeled other points in terms of fourths and asked, "what will be the two thirds of three fourths of this line

	(pointing to the region between 0 and 1)? Two fourths (pointing to the region between 0 and $\frac{2}{4}$)". She then labeled the portion of the number line from 0 to $\frac{2}{4}$ as $\frac{1}{2}$ and concluded, "so we can use different models to show the idea of multiplication of fractions".

Figure 6: Representations of $\frac{2}{3}$ of $\frac{3}{4}$ in Sam's class

Sam's primary goal seemed to be providing a variety of representations for this. The student might have drawn the first model using part-whole (recursive partitioning) or operator (split the region into three s and shade two) interpretation. Yet, because the student did not explain her thinking process fully and Sam did not follow up , it is hard to identify which interpretation was in use. Similarly, for the second drawing, although Sam often used a bar diagram for measurement situations, the student might have used the part-whole interpretation with recursive partitioning if she thought about the problem as part of a part. That the part already consisted of thirds blurs whether the student attended to the operator subconstruct when finding two thirds of three fourths. The last drawing seems to clearly use the measurement subconstruct given Lamon's (2007) definition of the notion (See Figure 1). However, Sam did not refer to the numbers in terms of their distances or measures from 0 since she asked what two-thirds of three-fourths would be pointing to the line segment between 0 and 1. If she considered this portion of the line segment as the whole that was partitioned, she might be attending to the part-whole interpretation with recursive partitioning.

In summary, both Eliot and Sam used visual representations to illustrate solutions to fraction multiplication problems. While doing so, Eliot relied on a single representation across problems whereas Sam used multiple representations for each problem. It remains unclear whether their use of visual representations also facilitated understanding of the different mathematical interpretations underlying fraction multiplication.

Discussion

Identifying the relationships between visual representations and mathematical interpretations was challenging in our study possibly because: (a) different interpretations of fraction multiplication could result in the same representation, and (b) instructors did not explicitly address which interpretations they were attending to as they represented fraction multiplication. For example, in using a subdivided area as both instructors did, whether they interpret fractions as part-whole or operator depends on the language they use to explain the representation and, in some cases, the order in which they subdivide the object. Making the steps clear could tie the fraction more closely to the interpretation or subconstruct. Another possibility is that using real contexts for fraction problems could lend meaning to the fractions that is absent in the abstract representations both of these instructors used.

One difficulty we encountered in analyzing these cases is that *representing a fraction* and *representing an operation with fractions* create different requirements for the teacher. Representing a single fraction using one of the subconstructs is relatively straightforward. Representing an operation, though, is not so easy. A subdivided area, as in Sam's pie diagram, can represent a part-whole fraction. But dividing each piece in half can be seen as creating smaller pieces (part-whole) or as operating on a single piece (operator). The language surrounding the representation as well as the choice of numbers in the multiplication problem is important for what idea the picture evokes for the student.

Does it matter? About this we have little evidence in this study, but previous work by Lamon (2007) suggests that it does matter. If K-8 students end up with a better understanding of and greater fluency with fractions by specifically learning about different interpretations of fractions, then it makes sense that teachers should themselves recognize these interpretations. We see in this case study, however, that the subconstructs of fractions can be intertwined during actual classroom practice. Our findings also indicate that it might be difficult to extract meaning from visual representations unless instructors clearly attend to the interpretations underlying those representations.

This study suggests several important areas for further research. In our view, it is especially interesting and important to understand more fully how explicit instructors of future teachers need to be about fraction interpretations and representations to equip their students – the future teachers of K-8 children – to teach fractions effectively.

Acknowledgements

This research is funded by the National Science Foundation (Grant No. 0447611). The authors wish to thank the two instructors who generously participated in this project and the other team members – Rachel Ayieko, Changhui Zhang, Andrea Francis, Rae-Young Kim, Jessica Liu, Jane-Jane Lo, Helen Siedel, and Sarah Young – who collected data and participated in discussions that made our analysis possible.

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