DEVELOPING PROSPECTIVE TEACHERS' KNOWLEDGE OF ELEMENTARY MATHEMATICS: A CASE OF FRACTION DIVISION

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Prospective elementary teachers must understand fraction division deeply to be able to teach this topic to their future students. This paper explores how two university instructors help prospective elementary school teachers develop such understanding. In particular, we examine how instructors teach the meaning of division, the concepts of unit, and the connections between multiplication and division.

Purpose of the Study

The National Mathematics Advisory Panel has identified "proficiency with fractions" as a major goal for k-8 mathematics education because "such proficiency is foundational for algebra and, at the present time, seems to be severely underdeveloped" (p. xvii). However, as acknowledged by the authors of *The Mathematical Education of Teachers* (Conference Board of the Mathematical Sciences (CBMS), 2001) and supported by prior research studies, many prospective and practicing teachers possess shallow understanding of fractions (e.g., Ball, 1990; Ma, 1999; Simon, 1993; Tirosh & Graeber, 1989), and some are convinced that "mathematics is a succession of disparate facts, definitions, and computational procedures to be memorized piecemeal" (p. 17, CBMS, 2001). This characterization stands in stark contrast to the depiction of mathematical knowledge needed for teaching that has arisen from research on the mathematical knowledge that teachers draw upon in the context of teaching. This research (cf., Ball, Thames & Phelps, 2008) suggests that prospective teachers need mathematical knowledge and skills beyond basic competency with the topics they intend to teach. They need, for example, to be able to give or evaluate mathematical explanations, and to connect representations to underlying mathematical ideas and other representations.

How can college mathematics courses help prospective elementary teachers develop the deep understanding of mathematics that they will need for their future teaching? In this paper, we analyze two sets of fraction division lessons for prospective elementary teachers to highlight both the content and nature of two different approaches to achieving this goal. We chose to focus on fraction division because of the well-known and well-documented struggles of U.S. prospective and practicing teachers with fraction division. For example, In Ma's study (1999), 20 of the 21 U.S. teachers were unable to come up with correct story problems for the given fraction division sentence 1 ¾ ÷ ½. The findings of this study provide paradigm cases to highlight the challenges of designing mathematics courses for prospective elementary teachers.

Theoretical Framework and Prior Study

Based on interviews with U.S. and Chinese elementary teachers, Ma (1999) proposed a 'knowledge package for understanding the meaning of division by fractions" that teachers should have as illustrated in the diagram below (from Ma, p. 77). In this paper, we focus on three specific aspects of this knowledge package, as suggested by the bolded objects in the diagram: the meaning of division for both whole numbers and fractions, the concepts of unit, and the properties and relationships among four basic operations.

Fishbein et al. (1985) identified two primitive models for division: a partitive model and a measurement model. For measurement division, one tries to determine how many times a given quantity is contained in a larger quantity. For partitive division, an object (or collection of objects) is divided into a given number of equal parts (or sub-collections), and the goal is to determine the quantity in (or size of) each part (or sub-collection). The "primitive" version of partitive division restricts the number of equal parts to a whole number, thus precluding division by a fraction, and reinforcing the idea that division makes smaller. Tirosh and Graeber (1989) found that partitive division was the dominating model held by U.S. prospective teachers, which led many of them to believe that in a division problem, the quotient must be less than the dividend, even though they could apply procedures to solve problems with divisor less than one correctly. This finding has prompted increasing attention to fraction division in the measurement context as well as calls for a modified interpretation of partitive division by capturing the idea of division as an inverse operation of multiplication. For example, Parker and Baldridge (2003) suggested thinking about 12 ÷ 2/3 as "12 is 2/3 of what?"

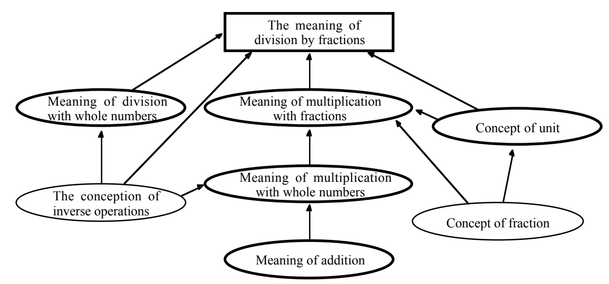

Fig. 3.2. A knowledge package for understanding the meaning of division by fractions
From Ma, 1999, p. 77.

The concept of unit in definitions and in operations is a key part of developing a deeper understanding for fraction division. For example, solving a measurement division problem such as "How many 2/3's are in 2?" requires the students to conceptualize "2/3" as a reference unit and interpret the "2" in terms of chunks of that particular unit: a process called "unitizing" by Lamon (1996). In the context of partitive division, such as when Parker and Baldridge (2003) suggest that students think of 12 ÷ 2/3 as "12 is 2/3 of what?" the unitizing process is more complex. In partitive division an object –the initial unit – is divided into a given number of equal parts, in this case 2/3 of a part. The goal is to determine the size of each part, a new unit, in this case 18. To solve a partitive division problem, a student must conceptualize the "unknown" quantity as both a unit itself and a fraction of a different unit.

Finally, the properties and relationships among operations (addition, subtraction, multiplication, and division) are needed when developing alternative algorithms (e.g. solving division word problems through repeated subtraction) or when explaining why the "flip and multiply" algorithm works.

Methods

This paper reports findings from the case study component of a large-scale research project that investigates mathematics content courses taken by prospective elementary teachers during their undergraduate education. We focus here on two of seven case studies in the larger study, the cases of Pat and Eliot. During the units on fractions, we videotaped, wrote observation notes, and collected artifacts from students and from the instructor. We interviewed the instructors to probe their ideas about teaching the course, and both instructors completed an extensive written survey about their teaching. As part of the larger project, students in theses courses took pre- and post-tests assessing their mathematical knowledge. Results of these tests suggest that both of these instructors were successful at teaching their students mathematics, producing among the highest gain scores of all 42 instructors in the larger study. (For additional information about the larger project and the pre-post-test results, see McCrory, 2009.)

Both Pat and Eliot taught at universities that prepare large numbers of future teachers in their respective states. They provided the greatest theoretical contrasts in their professional backgrounds and instructional approaches to fraction division among the seven participating instructors. Eliot was a new instructor who had received her Ph.D. in mathematics the previous year. This was the second time she taught this course and her instructional approach was a combination of lecture and individual seatwork. Pat was an experienced math instructor with a Ph.D. in mathematics education and several years' experience teaching high school math. He had taught this mathematics course for future teachers over 20 times. The majority of his class time was spent on a combination of small group work and students explaining and justifying their solutions in front of the class, interspersed with his comments, questions, clarifications, or explanations. He occasionally gave a prepared short (15 minute) lecture. Finally, the course taught by Eliot was a 3-credit math content course that met for 50 minutes three times a week, while the course taught by Pat was a 4-credit integrated content and methods course taught for 80 minutes twice a week.

Data from multiple sources for each instructor was compiled. Tabular materials chronicled the major goals and instructional events of each lesson as well as narratives containing initial memos about the research questions were generated to form the case study database for each participating instructor. The research team went through the videotaped lessons to identify the opportunities prospective elementary teachers had to develop deeper understanding of fraction division. Episodes that illustrated the development of a particular aspect of the knowledge package of fraction division were selected and transcribed for further comparative analysis.

Results

Our analysis on the fraction division lessons taught by the seven participating instructors uncovered a wide variety of approaches and emphases. The discussion of Eliot's and Pat's lessons helps illustrate such diversity. In the following we will first provide a summary of the main activities for each instructor's instruction of fraction division. Then we will discuss the main differences between these two different instructors using specific episodes from their lessons.

Table 1: Summary of Eliot's and Pat's lessons on fraction division

Eliot	2/29/08 (50 min.) • Model fraction division with pattern blocks using two hexagons as the whole. • Explain why the invert and multiply algorithm works. 3/03/08 (8 min.) • Discuss the patterns of fraction division when the divisor is smaller, equal or larger than one. • Discuss the patterns of fraction division when the divisor is smaller, equal or larger than the dividend. 3/05/08 (24 min.) • Use reasoning and logic to estimate the result of fraction division. • Review of fraction division with pattern blocks. 3/07/08 (8 min.) • Review of fraction division with pattern blocks.
Pat	4/10/08 (40 min.) • Model the solution of a (single) measurement fraction division word problem.

	• What number sentence can be used for solving this word problem? • Why is it a division problem (vs. a multiplication problem)? • Why is it hard for elementary students to connect their solution for a word problem to a number sentence? 4/15/08 (70 min.) • Model the solution of a (single) partitive fraction division word problem. • Compare and contrast the type of mathematical knowledge needed for solving this word problem with a number sentence vs. with a pictorial model. • Why can the same number sentence be used to represent both partitive and measurement division word problems? • Connect both fraction division word problems with whole number division problems. • Discussed the invert and multiply algorithm and ask students to think about why it works for both types of fraction division as homework.

As noted in the introduction the three key features of a deeper understanding of fraction division include: the meanings of division for both whole numbers and fractions, the concepts of unit, and the properties and relationships among four basic operations. In terms of the key fraction division concepts, Eliot's lessons were based exclusively on the measurement interpretation of division. Pat's students had opportunities to make sense of both measurement division and partitive divisions, and also spent considerable time to contrasting the two. In the context of measurement division, both instructors emphasized the process of unitizing, conceptualizing the divisor as the "unit" to represent the given quantity (dividend).). Eliot explained why the division algorithm worked by utilizing various properties of operations, while Pat facilitated his students' own discovery of the logic and reason behind this algorithm. Next we describe in detail episodes from each of the instructors.

Eliot' Lessons

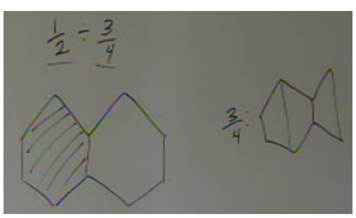

Figure 1: Eliot's board drawing for 1/2 ÷ 3/4

In Eliot's lessons, students were familiar with the "2-hexagon as the whole" model when using it to model operations with fractions. She expected that her students could move flexibly among representations and interpretations such as "1/2 ÷3/4", "How many ¾ in ½?" and "How many 3-trapezoids in one hexagon?" and the drawings associated with them. This was an approach that required developing an understanding of the model itself, as well as an understanding of the operation for division applied to fractions. Eliot modeled such processes for her students as shown in the following episode (see Figure 1 for Eliot's board drawing).

Eliot: Let's illustrate a half which we decided is one hexagon [Draws on board], and just to refresh our memory, three-fourths, we decided was three trapezoids. [Draws on board]. Now do I have an entire set of three trapezoids in my half? No. How much of three trapezoids do I have in my hexagon? [Some students responded two, others responded two-thirds.]

Eliot: Two-thirds. That's exactly it. I have two-thirds of three-fourths in one-half. So I have two-thirds of three trapezoids in two trapezoids. So I have two out of the three I was looking for in my shaded region. Two-thirds. (Transcript, 2/29/08)

In terms of the explanation of why the invert and multiply algorithm works, Eliot used 2/3 ÷ 5/7 as an example. As she wrote on the whiteboard, (Figure 2) she explained each step:

> What you are really doing when you flip and multiply is multiplying by one…. If I were to multiply by something over itself, I would be multiplying by the number one... So I want to multiply by seven-fifths over seven-fifths Have I changed a thing? No. …What is something multiplied by its reciprocal? One. So now all I have is two-thirds times seven-fifths divided by one. Well one is also the division identity. So guess what I have here? Two-thirds times seven-fifths. So what have I done effectively? Flipped it and multiplied. …This is why you can do that, because all you are really doing is multiply by the multiplicative identity. (Transcript, 2/29/08)

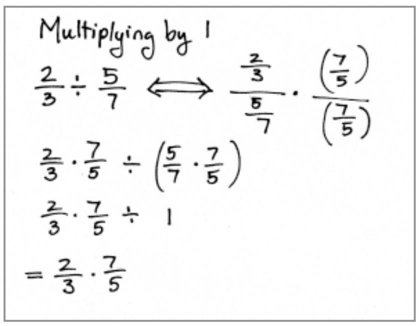

Figure 2: Eliot's explanation of "flip and multiply"

During all of her lessons on fraction division, Eliot designed her lessons around modeling with pattern blocks. She provided her students with clear, step-by-step explanations of the process and ample opportunities for them to practice on similar problems both in class where they could get additional support from her and as homework. She provided them with actual pattern blocks during class so that they could physically manipulate them. She acknowledged the struggle some of her students were having and re-visited this topic two more times, once after the quiz and once before the final exam, to address some of the common mistakes her students made. Eliot also modeled for her students how to use reasoning and logic to determine the reasonableness of their answers. She wanted her students know why the division algorithm works.

Pat's Lessons

Pat used the following word problem to discuss fraction division in the measurement context.

> *A batch of waffles requires ¾ of a cup of milk. You have two cups of milk. Exactly how many batches of waffles could you make?*

He gave students time to work on the problems in small groups (a mode of work that they were used to), and after about 30 minutes, asked the class what answers they got. Individual students gave answers – 2, 2 ¼ (later changed to 2 ½ after discovering a computation error), 2 2/3, 2 3/8. After some discussion the class agreed that there was enough milk to make 2 batches, and the computation 8/4-6/4=2/4=1/2 was carried out to get the answer 2 ½. One student who thought the answer was 2 2/3 batches was asked to explain his reasoning. He first wrote down 2/4 and ¾ and explained that 2/4 is 2/3 of ¾. He then drew the following diagram (Figure 3) while explaining his reasoning:

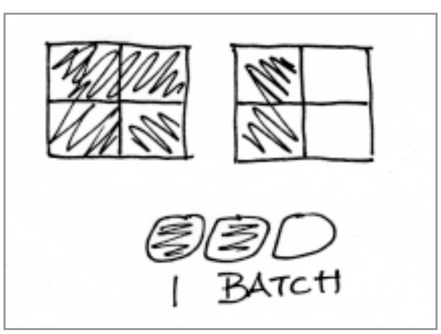

Figure 3: Student 1 waffle problem drawing

Student 1: This is two cups of milk [draws the two rectangles and then sub-divides each into four equal parts]. This is going to be batch number one right here [shades three parts of the rectangles, angle to the right]. And then batch two would be this [shades another three parts, angle to the left]. You got two boxes left. So there would be two boxes left. You need three boxes for a batch, so we have two boxes, we need three for a batch so we are only going to fill in two of these boxes [draws the three circles and shades two of them], 'cause this is one batch right here [points to the three circles and writes "1 BATCH" underneath them]. It's every three boxes. We have two left, which means there is two thirds of the last batch. So we have one batch right here [points to first three rectangle parts shaded], two batches [points to the next three], and two thirds of the last batch. ...the number two is two-thirds of three. ... I changed the wholes or whatever…. two-quarters is two-thirds of three-quarters. (Transcript 4/10/08)

Pat re-iterated the entire explanation and emphasized the point of "2 is 2/3 of 3, so you have 2/3 of a batch" which he wrote on the board. He also wrote "2/3 of a batch" and "½ of a cup" on the board for the shaded two of the three circles. He then asked the class what the student meant by changing the whole.

Student 2: Every time you make a batch of waffles, your whole or what is left over changes.

Student 3: You are changing from the whole as being the cup to the whole as being one batch of waffles.

Pat: So a cup is a whole and a batch is a whole… rather than writing 2 ½, you have 2 batches and ½ a cup of milk left over…. These two boxes [pointing to the bottom of the students' drawing] have double meaning. (Transcript, 4/10/08)

Instead of providing his students with a representation like the 2-hexgon, Pat asked them to generate their own drawings. He continued to push his students on being explicit about their explanation. In the process, he provided ample opportunities for his students to make connections among multiple representations: the story context, the drawings, the words, and the number sentences. His students were comfortable with being pushed and they also started to push each other for clear explanations or other their own elaboration without being prompted.

Pat introduced the question of why "flip and multiply" works after discussion of the waffle problem. Many students settled on the number sentence "2 x 4/3" for the waffle problem, but Pat pointed out that 4/3 was not a number in the problem (a requirement for an acceptable number sentence). One student offered an explanation using algebra, ¾ • x = 2 so x = 2 • 4/3. Pat asked for another justification that would work in their teaching, pointing out that an algebraic equation was beyond the comprehension of elementary students. Another student proposed thinking of 4/3 as the number of batches that could be made with one cup of milk. Pat delayed the rest of the discussion until the next class, during which the class worked on an additional contextualized division-by-fractions problem, this one a partitive problem:

You have 2 cups of flour to make some cookies. This is ¾ of what you need for one full recipe. How many cups of flour are needed for a full recipe?

Pat again asked students to work on the problem in groups, then to share and explain their solutions at the board. At the end of the second class, he asked students to use the pictorial representations of the two problems to figure out why the invert and multiply algorithm makes sense as a homework problem.

Following the principles of Cognitive Guided Instruction (Carpenter, et. al, 1999), Pat's lessons on fraction division were built upon story problems embedded in daily contexts. His students were encouraged to develop their own solution methods and models to explain their reasoning. Not apparent in the short episode discussed earlier were attempts both Pat and his students made to compare and contrast different models based on different solution methods of the same given problem.

Discussion

In this paper, we described the opportunities to develop deeper understanding of fraction division offered by two mathematics instructors in their courses. In terms of content, Eliot's lessons addressed topics that were not discussed in Pat's class, while Pat's lessons went deeper in connecting the meanings of whole numbers, fractions, multiplication and divisions through contextualized problems. Furthermore, Pat insisted on developing language and representations accessible to elementary students, while Eliot used concepts and terminology that would not be familiar to elementary school students.

Some of these differences are surely the result of the difference in course purposes: Eliot's a mathematics content course; Pat's an integrated content & methods course. They may also be a result of the difference in instructor backgrounds: Eliot a mathematician and Pat a mathematics educator and former high school math teacher. The effectiveness of these different curriculum models, math first then methods vs. integrated math and methods, is beyond the scope of the current study, as is the effectiveness of their very different approaches to teaching these ideas. We can observe, however, some differences in the mathematics of the lessons and provide conjectures about what these differences might mean for future teachers.

One big difference is the representations and how they were used. Eliot relied on pattern blocks and modeled reasoning with pattern blocks for her students. Pat encouraged his students to generate their own diagrams and expected them to use the context of the story problems to support their explanations. Both approaches get at the meaning of fractions and require moving flexibly across representations, which Lamon (2007) noted as a key part of fraction division understanding.

We also noticed a difference in the level of abstraction different representations demanded. For example, in Eliot's case, students need to be able to relate the actual physical blocks with the fraction quantities each block represents. Although the manipulatives are "real", the association of the block with the fraction is abstract and requires learning to connect the two. In Pat's case, the students need to create diagrams that connect with fraction quantities and the corresponding unit. In this case, the connections have meaning outside of the realm of mathematics, and may not be experienced as abstract. They also needed to attach each number and picture to something in the context of the word problem. The uses of both manipulatives (e.g. pattern blocks, fraction bars) and student-generated diagrams have been the primary focus of prior research investigations, and the findings have highlighted the complexity of making such contexts meaningful in elementary classrooms (e.g. Olive, 2000). The question, "How might these different uses of representations affect the development of deeper understanding of fraction division among prospective elementary teachers?" is worth pursuing.

The goals of mathematics courses specifically designed for prospective teachers should go beyond K-12 mathematics in order to distinguish themselves from mathematics courses for non-teachers. Both instructors did this. Eliot takes the students to mathematical explanations of the underlying mathematics that would not be appropriate for elementary students (e.g., the "flip and multiply" explanation) but, if successful, serves to provide the future teachers with deeper understanding of why the algorithm works. Pat's use of story problems requires students to understand why the problem is division and write number sentences for the problems they are learning. They give public explanations for their reasoning, thus teaching each other.

In this paper, we analyzed two sets of fraction division lessons for prospective elementary teachers and highlighted how two different representational contexts were used to achieve this goal. Even though mathematics courses for prospective elementary teachers are just a small component of the professional development continuum, these courses provide a common context to reach a large number of prospective elementary teachers. Thus it is important that we continue to explore how such classes are taught and how instructors choose and use representations to help future teachers learn mathematics.

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References

Ball, D.L. (1990). The mathematical understandings that prospective teachers bring to teacher education. *The Elementary School Journal, 90*(4), 449-466.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59, 399-407.

Carpenter, T. A., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). *Children's mathematics: Cognitively guided instruction*. Portsmouth, NH: Heinemann.

Conference Board of the Mathematical Sciences (2001). *The mathematical education of teachers* (Vol. 11). Providence, RI: American Mathematical Society American Mathematical Society & Mathematical Association of America.

Fischbein, E., Deri, M., Nello, M., & Marino, M. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education, 16*, 3-17.

Lamon, S. J. (1996). The Development of unitizing: Its role in children's partitioning strategies. *Journal for Research in Mathematics Education*, 27, 170-93.

Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In *Handbook of Research on Mathematics Education,* 629-667.

Ma, L. (1999). *Knowing and learning mathematics for teaching: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum Associates.

McCrory, R. (2009). *ME.ET progress report*, available online at http://meet.educ.msu.edu/reports/Advisory Board Report_010209.pdf

Olive, J. (2000) Computer Tools for Interactive Mathematical Activity in the Elementary School. *International Journal of Computers for Mathematical Learning*. 5, 241-262.

Parker, T. H. & Baldridge, S. J. (2003). Elementary Mathematics for Teachers, Okemos, MI: Sefton-Ash Publishing.

Simon, M. (1993). Prospective elementary teachers' knowledge of division. *Journal for Research in Mathematics Education, 24,* 233-254.

Tirosh, D., & Graeber, A. O. (1989). Preservice elementary teachers' explicit beliefs about multiplication and division. *Educational Studies in Mathematics, 20*, 79–96.